Tracking Error Volatility Optimization and Utility Improvements

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ABSTRACT

The Markowitz (1952, 1959) portfolio selection problem has been studied and applied in many scenarios. However, for delegated managers it is generally accepted that rather than choosing portfolios by optimizing in mean and variance, they choose by optimizing in mean and tracking error variance, i.e. the relative variance. This benchmark relative investing leads to necessarily sub-optimal portfolios, as outlined in Roll (1992). What is most important though is not that the portfolios are sub-optimal but whether the portfolios are better than the alternative, i.e. better than the portfolios that the principals could build themselves. In this paper I outline the conditions under which delegated managers increase the principal's utility. Assuming that the principal's best alternative is the benchmark, only under a very unlikely and specific scenario is there no opportunity for utility improvement. Additionally, the conventional practices of beta constraints, studied in Roll (1992), and tracking error volatility constraints, studied in Jorion (2003), assure utility improvements for the principal. If these constraints are implemented properly, they force the delegated manager to buy a more efficient portfolio than the benchmark. Thus, even though relative utility maximization is sub-optimal, if the delegated manager is more skillful than the principal, delegated portfolio management is still likely preferred to naively holding the benchmark.

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I. Introduction

Delegated contracting in investment management is the most popular and predominant form of management in the modern investment industry. This delegated portfolio management creates a classical incentive problem where the additional incentives that affect the agent, the delegated manager, are not completely consistent with the incentives of principal. It is generally accepted and consistently asserted through theory that this delegated incentive leads to sub-optimal, less-good, portfolios. This is particularly true when comparing the portfolio that the principal would build, given the same level of expertise and opportunity, to the portfolio the agent would build given the delegated incentives. If the portfolio choice problem is thought of as an optimization problem, then it is not difficult to imagine that the delegated incentives create binding constraints that can at best generate portfolios that are equally preferred to the unconstrained choice. Most likely the constraints will lead to sub-optimal portfolios. This sub-optimality is often considered as borderline irrational in the literature and is used as an argument either for passive investment management or for the inefficiency of using benchmarks portfolios as performance measures. However, given the pervasiveness of delegated contracting in the investment industry, it is difficult to reconcile how this structure has lasted for so long (and become more popular) given the apparent inefficiency. The answer is that we need to ask the question differently: assuming principals have less skill and opportunity than agents, do the predominant delegated incentives lead to better portfolios than the principals could build themselves?

Portfolio theory and asset pricing were developed under the assumption that the agent in the delegated relationship is a perfect agent whose incentives are perfectly aligned with those of the principal. Seminal work such as Markowitz (1952), Sharpe (1964), and Lintner (1965) paved the path for most modern investment research but like most foundational research, simplifying assumptions are made that potentially distort reality. It is evident that researchers and practitioners alike were concerned with the delegated incentive problem from the early life of investment theory but not many had done work critiquing how the relative incentive affected the theoretical models. An early example of an influential paper that discusses the delegated portfolio management situation in depth is Treynor and Black (1973). Although this paper is much more about reconciling inefficiency in markets, exploited through security selection, with an equilibrium look at an efficient market, the structure of how researchers approach the delegated incentive problem derives from Treynor and Black's approach in considering relative security selection versus the market portfolio, a benchmark.

One of the earliest criticisms and direct recognitions of the unrepresentativeness of applying Markowitz mean-variance space, and subsequently the CAPM, to delegated portfolio management came from Professor Sharpe himself in Sharpe (1981). Sharpe hypothesizes that it is highly unlikely that mean-variance efficiency can be obtained in a decentralized portfolio management setting where the investment decision-making responsibilities are delegated to an agent, in particular multiple agents. He proposes the foundation of a delegated incentive problem in multiple external agents and is skeptical of an optimal solution. Elton and Gruber (2004) solve the problem, albeit with some very narrow constraints, suggesting that delegated portfolio management could reach optimality. However, Bisburgen, Brandt, and Koijen (2008) revisits the problem relaxing the assumption about the certainty of the agents' risk appetites and concludes that serious inefficiency exists without this assumption. Blake et. al. (2013) uses the BBK framework and applies it to the delegated relationships in the pension industry and, among many other things, shows that the delegated incentive is pervasive in professional investment management.

Although Sharpe (1981) may have motivated the study of delegated portfolio management, another seminal paper in the area is Bhattacharya and Pfleiderer (1985). This paper set the stage for studying the delegated incentive as a principal/agent problem. In this paper they model a utility relationship between the principal and the agent. Their model implies the same conclusions of the other research, that it is unlikely that the delegation can reach optimality in utility. A direct follow-up to this work, Admati and Pfleiderer (1997), assumes that the motivating factor behind the delegated incentive involves a conditional optimization and a benchmark that is the solution to the Markowitz optimization problem given the limited set of information. Their conclusion is again consistent with the other literature on the subject; the conditional optimization leads to necessary sub-optimality in portfolio choice.

This is the same conclusion reached in Roll (1992) albeit by a slightly different construction. Roll builds a framework much more closely related to Markowitz (1952) but instead of investment managers optimizing in mean/variance space, they optimize in mean and tracking error variance, the relative variance. This leads to a frontier, the TEV (Tracking Error Volatility) frontier that passes through the benchmark portfolio. The dependence on a benchmark as a relative incentive is the connection between Admati and Pfleiderer (1997) and Roll (1992). The overriding implication of Roll's construction is that agents optimize utility in mean and tracking error volatility, not in mean and variance. If we assume that the principal derives utility in mean-variance space and that the relative incentive causes the delegated agent to derive utility in mean-TEV space, then the broad implication of Roll's TEV frontier optimization is that we need to study the transformation of the agent's utility from a relative optimization to the principal's utility in absolute optimization.

If, for example, we assume that the portfolio used to benchmark the external manager is the principal's best option, then we can consider the principal's utility curve which passes through that benchmark to be the highest utility the principal can obtain given his own skill and information. The delegation to an external manager, as long as the principal's utility curve is not tangent to his opportunity set, the TEV frontier, at the benchmark, necessarily has a utility increasing deviation for the principal as long as the agent has better information or more skill than the principal and proper constraints are applied. The contribution of this essay is in directly extending Roll's framework by recognizing that the relative optimization problem can produce higher utility for the principal. Although it is true that the principal, given all the information and skill of the agent, would build a different and better portfolio, except for a very specific and unlikely case, the agent can still create higher utility for the principal than the default benchmark, even given the inefficiency of relative optimization.

The closest application of a utility problem directly to Roll's TEV frontier is in Bertrand (2010). In this paper he considers the problem of a fixed risk aversion constraint in mean/variance and its ability to generate preferred portfolios. My assumption is slightly different that Bertrand's in that managers cannot, in my framework, be constrained on risk aversion in mean and variance but only in risk aversion in mean and TEV. Although these spaces are related, they are different enough that they lead to very different conclusions. Based on my assumptions, Bertrand's iso-risk aversion curves are the path that a principal would like to follow, not the path the delegated manager actually follows. In section II, I look at the space of mean-TEV and translate it back to mean-variance. It is a direct application of Roll's construction with a relative utility overlay. The path an unconstrained relative optimizer would follow is far inferior to the efficient frontier and equivalent to the TEV frontier at the agent's portfolio choice given changes in the agent's risk aversion level. However it is still likely that this path increases utility for the principal, at least over some controllable range of possibilities. Section III considers the relationship of principal utility in the space of mean and variance given the principal's best alternative, investing in the benchmark, to the TEV frontier in mean-variance. I show that the delegated manager can almost universally increase the principal's utility.

Another important paper that stems directly from Roll (1992) is Jorion (2003), in which Jorion considers the frontier in mean variance space given a constraint to tracking error variance. This is an important problem in practice because in examples of real world delegated investment management, oftentimes external managers are constrained by a tracking error bound. In a similar fashion to Roll, Jorion concludes that the tracking error bound most likely incentivizes agents to take more variance than the benchmark in portfolio selection and that this constraint should be used with caution when applied to external managers. Using my relative utility framework, I show in section IV of this paper that when given only a tracking error constraint, principals can control the risk level of the external manager to guarantee that the portfolio selected by an agent will increase the principal's utility. If implemented properly, the tracking error constraint placed by principals on external managers is rational.

Another pervasive constraint applied to external managers in the industry is the constraint related to style drift. We can calculate beta relative to the underlying benchmark and consider deviations from factor sensitivity of 1 to be deviations due to style drift. Although this constraint is controversial in the industry, it is necessary for many of the theoretical conclusions in active portfolio management, most notably the Fundamental Law of Active Management, which is summarized well in Grinold and Kahn (2000). Roll (1992), in addition to deriving the TEV frontier, also considers a number of other problems, one being constraining benchmark relative beta. His conclusion is that the beta constraint generates a necessarily superior frontier to the TEV frontier in the region over which external managers are likely to optimize. I look at this beta constraint in the context of my relative utility overlay in section V and conclude much the same with similar caveats. However, the most interesting result of analyzing the beta constraint is that it always has the potential to increase the principal's utility. There are a couple scenarios under the tracking error constraint that, however unlikely, could cause the delegated manager to act against the interests of the principal. As long as the benchmark is above the global minimum variance portfolio in return, the beta constraint always has the potential to increase principal utility, particularly if the benchmark has a higher expected return that the minimum variance portfolio. Thus, just like the tracking error constraint, constraining external managers on beta is also rational.

The beta constraint itself is not enough to guarantee a superior portfolio from delegated management either. It is quite possible for the external manager, given low enough levels of risk aversion, to build a portfolio less efficient than without the constraint. Additionally, the tracking error constraint, however rational and possibly utility increasing, still lies on a frontier below that of a constrained beta frontier. It seems natural that the combination of these two constraints could force a delegated manager to invest on a frontier superior to the TEV frontier and additionally assure the principal that the external manager will not invest in a portfolio risky enough to erode utility. In section VI, I analyze the inclusion of both the tracking error constraint and the beta constraint in controlling a delegated agent. These constraints, if implemented properly, necessarily cause the delegated manager to choose a portfolio that increases utility for the principal. I use these constraints to develop a benchmarking strategy to force external managers to not only build a portfolio that increases utility, but to build a portfolio that maximizes the principal's utility given the agents opportunity set. Essentially, if the parameters are known, setting the tracking error and beta constraints to the levels of the desired portfolio, even the global optimal portfolio for the principal, incentivizes the agent to buy that exact portfolio. These two constraints are enough to pin the delegated manager to an exact location within investment opportunity set. They guarantee not only a unique but a superior portfolio.

The conclusion of this work is that if we believe delegated investment managers to have skill and expertise, then it is not only rational but preferred to delegate our investment management responsibilities to those individuals, even given that they are relative TEV optimizers. The big caveat here is the assumption that the external manager really does have the ability to outperform a benchmark. Two of the most popular risk control metrics in the industry, beta constraints and tracking error constraints, can be used to pin the external manager to an exact spot in mean-variance space. If the principal understands his own utility in this space, and has an idea of what the external manager's skill and capabilities are, then this spot is a necessary utility increase for the principal. The delegated portfolio choice, when delegated to a skillful agent and under the tracking error and beta constraints, is always better than the portfolio a principal could build himself.

II. The TEV Frontier and the Agent's Objective Function

The Markowitz mean-variance optimization problem involves minimizing the variance of a portfolio of securities. There are many well-known solutions to the Markowitz problem but perhaps the most popular is considering the problem as a quadratic programming problem. Below are the variable definitions and the problem's statement and solution. Matrices and vectors are in boldface.

w - $n \times 1$ the weight vector of the portfolio

Ω - $n \times n$ covariance matric (symmetric and positive definite)

r - $n \times 1$ vector of returns

1 - $n \times 1$ vector of 1's

The problem is to minimize the variance as a function of return given that the weight vector sums to 100%:

$$\min_{\mathbf{w}} \mathbf{w}'\boldsymbol{\Omega}\mathbf{w} \quad s.t.\ \mathbf{w}'\mathbf{1} = 1 \ \ and\ \mathbf{w}'\mathbf{r} = r \tag{1}$$

We can set up the Lagrangian as follows:

$$\mathrm{L}(\mathbf{w}, \lambda_1, \lambda_r) = \mathbf{w}'\boldsymbol{\Omega}\mathbf{w} - \lambda_1 (\mathbf{w}'\mathbf{1} - 1) - \lambda_r (\mathbf{w}'\mathbf{r} - r) \tag{2}$$

Next we differentiate and set the results to **0**:

$$2\boldsymbol{\Omega}\mathbf{w} - \lambda_1\mathbf{1} - \lambda_r\mathbf{r} = \mathbf{0}$$

$$\mathbf{w}'\mathbf{1} = 1 \tag{3}$$

$$\mathbf{w}'\mathbf{r} = r$$

Solving this system for **w** yields the following: (notation such as [**1** **r**] is an $n \times 2$ augmented matrix where the first column is all 1's and the second is the return vector; also, the last matrix in this notation is a 2×1 vector with the constants 1 and r):

$$\mathbf{w} = \boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r}]\left[[\mathbf{1} \quad \mathbf{r}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r}]\right]^{-1}\begin{bmatrix} 1 \\ r \end{bmatrix} \tag{4}$$

This weight vector can be plugged back into variance function and the parabola of the minimum variance set, i.e. the efficient frontier, can be expressed with the variance of the portfolio, σ^2, as a function of the return constant, r. In essence, this is the equation for the efficient frontier.

$$\sigma^2 = \mathbf{w}'\boldsymbol{\Omega}\mathbf{w} = \begin{bmatrix} 1 \\ r \end{bmatrix}'\left[[\mathbf{1} \quad \mathbf{r}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r}]\right]^{-1}\begin{bmatrix} 1 \\ r \end{bmatrix} \tag{5}$$

The Tracking Error Variance or TEV Frontier from Roll (1992) can be derived similarly but with a slightly different objective function. If the weight vector for the benchmark is denoted as **b**, then Roll's framework requires minimizing tracking error variance by choosing **w**. The problem proceeds similarly but with a slight complication because of the differenced weight vector (**w** – **b**). Below is a statement of the problem and the subsequent Lagrangian:

$$\min_{\mathbf{w}} T^2 = \min_{\mathbf{w}} (\mathbf{w} - \mathbf{b})'\boldsymbol{\Omega}(\mathbf{w} - \mathbf{b}) \quad s.t. \ \mathbf{w}'\mathbf{1} = 1 \ \ and \ \mathbf{w}'\mathbf{r} = r \tag{6}$$

$$\mathrm{L}(\mathbf{w}, \lambda_1, \lambda_r) = (\mathbf{w} - \mathbf{b})'\boldsymbol{\Omega}(\mathbf{w} - \mathbf{b}) - \lambda_1 (\mathbf{w}'\mathbf{1} - 1) - \lambda_r (\mathbf{w}'\mathbf{r} - r) \tag{7}$$

Below is the system of simultaneous equations to be solved. It is the same as the original problem but with the extra term, $2\boldsymbol{\Omega}\mathbf{b}$:

$$2\boldsymbol{\Omega}\mathbf{w} - 2\boldsymbol{\Omega}\mathbf{b} - \lambda_1\mathbf{1} - \lambda_r\mathbf{r} = \mathbf{0}$$

$$\mathbf{w}'\mathbf{1} = 1 \tag{8}$$

$$\mathbf{w}'\mathbf{r} = r$$

In a similar form to what was derived for the variance minimization, in the case of tracking error variance we choose the following weight vector where $r_b = \mathbf{b}'\mathbf{r}$.

$$\mathbf{w} = \mathbf{b} + \boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r}]\left[[\mathbf{1} \quad \mathbf{r}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r}]\right]^{-1}\begin{bmatrix} 0 \\ r - r_b \end{bmatrix} \tag{9}$$

This weight vector can be plugged bask into the variance function and an expression for the parabola of the TEV frontier as a function of r can be obtained. If we define $\sigma_b^2 = \mathbf{b}'\boldsymbol{\Omega}\mathbf{b}$ then:

$$\sigma^2 = \mathbf{w}'\boldsymbol{\Omega}\mathbf{w} = \sigma_b^2 + \begin{bmatrix} 0 \\ r - r_b \end{bmatrix}'\left[[\mathbf{1} \quad \mathbf{r}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r}]\right]^{-1}\begin{bmatrix} 0 \\ r - r_b \end{bmatrix} \tag{10}$$

$$+ 2\begin{bmatrix} 1 \\ r_b \end{bmatrix}'\left[[\mathbf{1} \quad \mathbf{r}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r}]\right]^{-1}\begin{bmatrix} 0 \\ r - r_b \end{bmatrix}$$

The relationship between the efficient frontier from (5) and the TEV frontier from (10) is illustrated and analyzed thoroughly in Roll (1992). Roll's primary conclusion is that the TEV frontier is less optimal than the efficient frontier whenever the benchmark portfolio is not on the efficient frontier. This is illustrated in Figure 1, Panel A. The TEV frontier is to the right of the efficient frontier because optimization on relative tracking error variance rather than absolute variance is less efficient. The horizontal axis in this figure is standard deviation. However, it is probably more useful to look at this diagram with tracking error volatility on the horizontal axis. This graph, with tracking error on the horizontal axis, is depicted in Figure 1 panel B and the equations for the curves are derived as follows.

First recall that the tracking error variance, T^2, is defined as follows:

$$T^2 = (\mathbf{w} - \mathbf{b})'\boldsymbol{\Omega}(\mathbf{w} - \mathbf{b}) \tag{11}$$

For the TEV frontier, we simply take the weight vector from (9) and move **b** to the left hand side.

$$(\mathbf{w} - \mathbf{b}) = \boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r}]\left[[\mathbf{1} \quad \mathbf{r}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r}]\right]^{-1}\begin{bmatrix} 0 \\ r - r_b \end{bmatrix} \tag{12}$$

Plugging this vector into the tracking error variance equation yields the following parabola where T^2 is a function of r. This is the TEV frontier in the space of mean and tracking error variance rather than in mean and variance. This is, not coincidentally, just one of the terms from equation (10).

(Insert Figure 1)

$$T^2 = \begin{bmatrix} 0 \\ r - r_b \end{bmatrix}' \left[[\mathbf{1} \quad \mathbf{r}]' \boldsymbol{\Omega}^{-1} [\mathbf{1} \quad \mathbf{r}] \right]^{-1} \begin{bmatrix} 0 \\ r - r_b \end{bmatrix} \tag{13}$$

Given the weight vector for the efficient frontier from equation (4), we can adjust it by differencing it with the benchmark as follows. Note that a **b** was just subtracted from both sides.

$$(\mathbf{w} - \mathbf{b}) = \boldsymbol{\Omega}^{-1} [\mathbf{1} \quad \mathbf{r}] \left[[\mathbf{1} \quad \mathbf{r}]' \boldsymbol{\Omega}^{-1} [\mathbf{1} \quad \mathbf{r}] \right]^{-1} \begin{bmatrix} 1 \\ r \end{bmatrix} - \mathbf{b} \tag{14}$$

Plugging this vector into the tracking error variance equation yields the following parabola. This is the equation for the efficient frontier when plotted in the space of tracking error variance.

$$T^2 = \begin{bmatrix} 1 \\ r \end{bmatrix}' \left[[\mathbf{1} \quad \mathbf{r}]' \boldsymbol{\Omega}^{-1} [\mathbf{1} \quad \mathbf{r}] \right]^{-1} \begin{bmatrix} 1 \\ r \end{bmatrix} + \sigma_b^2 - 2 \begin{bmatrix} 1 \\ r_b \end{bmatrix}' \left[[\mathbf{1} \quad \mathbf{r}]' \boldsymbol{\Omega}^{-1} [\mathbf{1} \quad \mathbf{r}] \right]^{-1} \begin{bmatrix} 1 \\ r \end{bmatrix} \tag{15}$$

The mean variance frontier, when mapped to tracking error variance space is to the right of the TEV frontier just like the TEV frontier is to the right of the efficient frontier when in the space of absolute variance. This is the phenomenon illustrated in Figure 1, Panel B.

One of the most important implications of the TEV frontier is the implication about how delegated managers choose portfolios given a relative performance incentive. That is, they optimize return relative to tracking error rather than variance or standard deviation. This implication also directly asserts that agents will optimize a utility function that is parameterized with tracking error volatility as the risk measure rather than standard deviation. No matter the exact structure of the utility relationship, utility and preference in relative space can be thought of exactly like it is thought of in absolute space. The preference set is a closed, convex set and increases to the "northwest" just as it does in mean-variance. Also, the iso-utility curve is upward sloping with a non-negative second derivative. The iso-utility curve associated with optimizing utility in relative space sits exactly tangent to the TEV frontier in relative space and this is illustrated in Figure 1 Panel B. For the purposes of illustration, I derive all of the utility relationships using quadratic utility but it should be evident that any properly formed utility function generally obeys all of the rules of the quadratic utility for the purposes of this essay.

Suppose the utility function for the agent is as follows where θ is the coefficient of risk aversion:

$$U = r - \theta\, T^2 = \mathbf{w}'\mathbf{r} - \theta(\mathbf{w}-\mathbf{b})'\boldsymbol{\Omega}(\mathbf{w}-\mathbf{b}) \tag{16}$$

The problem is to maximize utility subject to the constraint on the weight vector.

$$\max_{\mathbf{w}} U = \max_{\mathbf{w}}\ \mathbf{w}'\mathbf{r} - \theta(\mathbf{w}-\mathbf{b})'\boldsymbol{\Omega}(\mathbf{w}-\mathbf{b}) \quad s.t.\ \mathbf{w}'\mathbf{1} = 1 \tag{17}$$

$$\mathrm{L}(\mathbf{w},\, \lambda_1) = \mathbf{w}'\mathbf{r} - \theta(\mathbf{w}-\mathbf{b})'\boldsymbol{\Omega}(\mathbf{w}-\mathbf{b}) - \lambda_1\,(\mathbf{w}'\mathbf{1} - 1) \tag{18}$$

Differentiating the Lagrangian and finding the critical value yields the following solution for the optimal weight vector:

$$\mathbf{w} = \frac{1}{2\theta}\boldsymbol{\Omega}^{-1}\left(\mathbf{r} - \frac{\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\mathbf{1}\right) + \mathbf{b} \tag{19}$$

The portfolio representing this optimal weight vector is depicted in Figure 1, Panels A and B, with two different levels of risk aversion. In Panel A, it is necessarily along the TEV frontier above the benchmark portfolio and it is notable that it not possible for this portfolio to be on the efficient frontier unless the benchmark is also on the efficient frontier. As the agent's risk aversion coefficient decreases, the agent chooses a portfolio further and further up the TEV frontier. In Panel B, this portfolio is the optimal allocation and along the TEV frontier. Also depicted in Panel B is the iso-utility curve associated with this portfolio's level of utility. It can be back-solved simply by rearranging the utility function from (16) and plugging in the weight vector from (19), and it is expressed as follows:

$$
\begin{aligned}
r = U + \theta\, T^2 &= \mathbf{w}'\mathbf{r} - \theta(\mathbf{w}-\mathbf{b})'\boldsymbol{\Omega}(\mathbf{w}-\mathbf{b}) + \theta\, T^2 \\
&= r_b + \frac{1}{4\theta}\left(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{r} - \frac{(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1})^2}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right) + \theta\, T^2 \qquad (20)
\end{aligned}
$$

Delegated managers that are incentivized by a relative return incentive will optimize along the TEV frontier and maximize relative utility in the space of mean and tracking error volatility. This gives rise to the utility relationship as depicted in (20). This is the agent's iso-utility curve and in general it is inconsistent with the process of maximizing utility in mean/variance space, the space in which the principal derives utility.

Table 1 calculates the quadratic utility deviations for the principal under differing scenarios. Panel A depicts the utility increase or decrease for the principal from unconstrained delegated contracting. The delegated performance incentive has the potential to increase utility for the principal but this is far from certain. Given the levels of utility I use in the figures for this paper, the utility decreases by 5.15% through delegation. Panel B shows the utility depreciation from the principal's global optimal portfolio. This is necessarily non-positive. In this paper, the agent chooses a portfolio that 9.25% less optimal in utility than the principal's global optimal. This example is consistent with what was shown above. Although the agent has the potential to increase utility above that of the benchmark, unconstrained there is no guarantee for utility improvement.

(Insert Table I)

III. The Principal's Utility and Improvements along the TEV Frontier

As is conventional in portfolio management literature, I assume that the principal investor derives utility in mean-variance space. Thus his goal is to maximize utility subject to the constraint of the mean-variance efficient frontier. The premise of this essay is that principals delegate portfolio management responsibilities because they lack the ability or opportunity to build efficient portfolios themselves. The principal expects the delegated manager to build a portfolio better than the principal could build. Therefore the most logical basis for determining whether an agent improves upon the principal's utility is to compare the agent portfolio choice to whatever the principal could build given his limited skill level and opportunity set. If we suppose that the principal benchmarks the external manager against his best alternative, and incentivizes the agent relative to this benchmark (implicitly or explicitly), then this necessitates that the principal's utility curve pass through that benchmark portfolio. In the previous section we showed how the agent derives utility to create the TEV frontier. The relative positioning and interaction between these two curves, the TEV frontier and the principal's iso-utility curve, is what determines whether the agent can indeed improve the principal's utility given the delegated performance incentive.

Just as before when discussing the agent's utility, as long as the principal's utility is well formed, upward sloping with a non-negative second derivative in mean-variance space, the exact expression of the utility relationship is irrelevant to the conclusions of this essay. Utility is increasing for the principal to the "northwest" and the preference set is closed and convex. Notably, the opportunity set underneath the TEV envelope in mean-variance space is also a closed convex set. As discussed previously, these two sets intersect at least once, at portfolio **b**, which lies on the boundary of both sets. Unless the TEV frontier and the utility curve are exactly tangent at the benchmark portfolio, then the opportunity for utility improvement exists because in this framework there would be an overlap between the sets. There are four interesting cases of how these two sets could intersect, and these cases affect how the principal should constrain the external manager to assure utility improvement. The first three cases involve the slope of the sets at the intersection point, **b,** assuming **b** is above the minimum variance point. The slope of the utility curve can be steeper, flatter, or the same as the TEV frontier at this point. The fourth case is when the intersection happens given **b** is below the minimum variance portfolio. These cases are analyzed heuristically in Figure 2 and analytically with derivations in quadratic utility.

The steepness of the utility relationship is equivalent to the level of risk aversion. The higher the coefficient of risk aversion is, the steeper is the utility relationship in mean and variance. Below is an example of the utility function of the principal and note that is it parameterized in mean and variance. θ is the coefficient of risk aversion. And once again, quadratic utility is used in this example for simplicity in expression. In general, all of the assertions and conclusions in this essay are true no matter the exact form of the utility function. Thus, suppose principals maximize utility by choosing **w** given the following relationship:

$$U = r - \theta\,\sigma^2 = \mathbf{w}'\mathbf{r} - \theta\mathbf{w}'\mathbf{\Omega}\mathbf{w} \tag{21}$$

If the principal had the ability and opportunity to maximize this utility function given the universe of all investment opportunities, he would choose a portfolio by maximizing this function, unconstrained as follows:

$$\max_{\mathbf{w}} U = \max_{\mathbf{w}} \mathbf{w}'\mathbf{r} - \theta\mathbf{w}'\mathbf{\Omega}\mathbf{w} \quad s.t. \;\mathbf{w}'\mathbf{1} = 1 \tag{22}$$

$$\mathrm{L}(\mathbf{w}, \lambda_1) = \mathbf{w}'\mathbf{r} - \theta\mathbf{w}'\mathbf{\Omega}\mathbf{w} - \lambda_1 (\mathbf{w}'\mathbf{1} - 1) \tag{23}$$

Differentiating the Lagrangian and finding the critical value yields the following result.

$$\mathbf{w} = \frac{1}{2\theta}\mathbf{\Omega}^{-1}\left(\mathbf{r} - \frac{\mathbf{r}'\mathbf{\Omega}^{-1}\mathbf{1} - 2\theta}{\mathbf{1}'\mathbf{\Omega}^{-1}\mathbf{1}}\mathbf{1}\right) \tag{24}$$

This is the weight vector of the portfolio that the principal would ideally like to hold given the agent's opportunity set. Applying this vector to the utility relationship in (21) reveals the maximum iso-utility curve given the constraint of the efficient frontier.

$$\begin{aligned} r = U + \theta\,\sigma^2 &= \mathbf{w}'\mathbf{r} - \theta\mathbf{w}'\mathbf{\Omega}\mathbf{w} + \theta\,\sigma^2 \\ &= \frac{1}{4\theta}\left(\mathbf{r}'\mathbf{\Omega}^{-1}\mathbf{r} - \frac{(\mathbf{r}'\mathbf{\Omega}^{-1}\mathbf{1} - 2\theta)^2}{\mathbf{1}'\mathbf{\Omega}^{-1}\mathbf{1}}\right) + \theta\,\sigma^2 \end{aligned} \tag{25}$$

This level of utility is practically unreachable however because of the principal's lack of ability and opportunity. I assumed earlier that the principal can select a portfolio **b** that maximizes utility given his constrained skillset. If this is the portfolio on which the principal measures and incentivizes the agent, then this is the same **b** from the TEV analysis in section II. The utility of portfolio **b** and the iso-utility curve associated with the utility of portfolio **b** are expressed as follows:

$$U = \mathbf{b}'\mathbf{r} - \theta\mathbf{b}'\mathbf{\Omega}\mathbf{b} = r_b - \theta\,\sigma_b^2 \tag{26}$$

$$r = U + \theta\,\sigma^2 = r_b - \theta\,\sigma_b^2 + \theta\,\sigma^2 = r_b + \theta\,(\sigma^2 - \sigma_b^2) \qquad (27)$$

Thus, we are concerned about the interaction between the curve from (27) and the curve from (10), the TEV frontier.

Figure 2, Panel A shows the most likely (or at least the most convenient) scenario for this relationship. In this case, the slope of the TEV frontier is steeper than the slope of the principal's utility curve at **b**. Recall from section II and also from Roll (1992) that an agent optimizing based on a relative incentive will choose a portfolio by moving along the TEV frontier up from the benchmark portfolio. In this case, since the utility curve has a flatter slope than the TEV frontier at the intersection point **b**, the preference set, the set above the iso-utility curve, overlaps with the opportunity set, the set underneath the TEV envelope, and every point in the intersection is a utility increase for the principal. In particular, as the agent's risk aversion level decreases, he differentiates from the benchmark and moves up the curve into this preferred space. Eventually however, the agent's risk aversion level could get so low that his optimization process pushes the portfolio back out of the preferred space. Therefore, if the agent is allowed to act on his own unconstrained utility, it is still likely that he will build a portfolio that decreases the principal's utility even though he could have increased it by buying a portfolio within the preferred region. I will discuss a method to constrain the agent in the next section but what should be evident at this point is that delegated portfolio management very likely could increase principal utility.

(Insert Figure 2)

Figure 2, Panel B shows the opposite scenario to Panel A. In this case, the principal's utility curve is steeper than the TEV frontier at **b**. Although a space still exists in the intersection of the preference set and the opportunity set, this space is on the wrong side of **b**. The process the agent follows when incentivized with the relative return inventive will push the portfolio choice away from the preferred region rather than into it. This may seem like a dire situation, but there is a simple transformation in this space that allows the relative incentive to continue to be used to the principal's advantage. First we should recognize that the principal's utility must have a flatter slope at **b** than the line segment that connects portfolio **b** to the origin. This is true only if we assume that the origin could be a potential long investment opportunity for the principal. If the line segment is not steeper than the utility curve then there exists a portfolio on the line segment with a higher utility than portfolio **b**, and this violates the assumption behind portfolio **b**. That is, portfolio **b** must be the highest utility portfolio available to the principal given his skill level and opportunity set. This condition assures us that this line segment intersects the TEV frontier below the set where the preference set intersects with the opportunity set. If we define this lower intersection point as point $\mathbf{b}_L$, then we have transformed the Case B problem into a Case A problem. If the delegated manager is now benchmarked against $\mathbf{b}_L$, then we are back into a situation where the delegated manager can create a utility increase under the right constraints. Care must be taken in this situation however because the principal needs to worry not only about the agent taking too much risk but also not taking enough risk. Deviations up the TEV frontier from $\mathbf{b}_L$ must cross a threshold before the preferred region is reached.

Figure 2, Panel C depicts the situation when the utility curve is exactly tangent to the TEV frontier at **b**. Given that the intersection is a single point, there is no region in which a preferred deviation would be made by a delegated manager incentivized by a relative return incentive alone. This truly is a dire situation for delegated management. In this scenario, it would be irrational to delegate portfolio management responsibilities to an unconstrained agent because the principal already holds the best possible portfolio given the delegated performance incentive. This situation is most comparable to the situation analyzed in Admati and Pfleiderer (1997). They find, among other things, that when a delegated manager is incentivized by a relative performance incentive, given an efficiently allocated benchmark, there is no possibility for utility improvement through delegation. This would indicate that the principal should choose not to delegate. However, in the Roll (1992) framework, which I am using in this essay, there is a way to constrain the external manager to necessarily improve utility even in this dire situation. This involves the beta constraint proposed by Roll and it is applicable to all three cases presented thus far. The beta constraint is analyzed in this paper in sections V and VI.

Figure 2, Panel D is the case when the benchmark portfolio, **b**, lies below the minimum variance portfolio and equivalently also below the minimum point on the TEV frontier. This is a special case of Case A. The reason it is special is because no matter the slope of the utility curve at **b**, deviations up from **b** along the TEV frontier always increase utility. As favorable as this situation seems, at least anecdotally, I consider this situation highly unlikely but not impossible. In order for this to occur, however, there would have to be massive inefficiency in the investment set available to the principal. Thus, although this case is intellectually curious, it is probably also practically impossible.

As depicted in Figure 2, it is highly likely that given the interaction between the principal's utility curve and the TEV frontier, that benchmarking and the implied relative return incentives could potentially create a utility increase for the principal. However, care must be taken in this relationship to ensure that the agent does not take too much risk as to exceed the region in which a preferred portfolio could be chosen. Thus at this point the framework is set to analyze the proposed constraints on external managers. How can we force the external manager to buy a portfolio within the preferred region?

IV. Constraining the Agent's Risk Appetite using a TEV Constraint

To control the behavior of external managers, other papers have assumed a direct constraint on agent risk aversion, such as Bertrand (2010). However, risk aversion is the parameter that defines the behavior of the individual and trying to forcibly constrain (or extend) an agent's risk level is akin to trying to constrain a law of nature. A principal could filter potential delegated managers based on his perceived notion of the agent's risk aversion but implicit in this filtering is a situation that creates more uncertainty, i.e. more risk, due to the potential for asymmetric information. Additionally, there is no good way to either ex-ante or ex-post measure an agent's risk aversion level. It can be proxied ex-post but would require a substantial amount of data. Practically however, if it could be done, constraining on agent risk aversion would peg the external manager to the optimal point for the principal along the TEV frontier. Constraining risk aversion would only allow the delegated manager to move so far up the frontier until the point where the principal's utility is optimized. But, why use the something so elusive when a conventional measure exists that is more tangible, measurable, and serves the same purpose: tracking error volatility.

Jorion (2003) considers the problem of maximizing (and minimizing) return given a tracking error constraint in mean-variance space. Equivalently we could minimize (and maximize) standard deviation or variance given a tracking error constraint. If the problem is set up in this fashion, on variance, it looks like the following:

$$\min_{\mathbf{w}} \mathbf{w}'\boldsymbol{\Omega}\mathbf{w} \quad s.t.\ \mathbf{w}'\mathbf{1} = 1\ \ \mathbf{w}'\mathbf{r} = r\ and\ (\mathbf{w}-\mathbf{b})'\boldsymbol{\Omega}(\mathbf{w}-\mathbf{b}) = T^2 \tag{28}$$

And the Lagrangian can be set up like this:

$$\begin{aligned} &\mathrm{L}(\mathbf{w}, \lambda_1, \lambda_r, \lambda_{T^2}) \\ &\quad = \mathbf{w}'\boldsymbol{\Omega}\mathbf{w} - \lambda_1(\mathbf{w}'\mathbf{1} - 1) - \lambda_r(\mathbf{w}'\mathbf{r} - r) \\ &\quad - \lambda_{T^2}((\mathbf{w}-\mathbf{b})'\boldsymbol{\Omega}(\mathbf{w}-\mathbf{b}) - T^2) \end{aligned} \tag{29}$$

Differentiating yields the following set of simultaneous equations.

$$\begin{gathered} 2\boldsymbol{\Omega}\mathbf{w} - \lambda_1\mathbf{1} - \lambda_r\mathbf{r} - \lambda_{T^2}(2\boldsymbol{\Omega}\mathbf{w} - 2\boldsymbol{\Omega}\mathbf{b}) = \mathbf{0} \\ \mathbf{w}'\mathbf{1} = 1 \\ \mathbf{w}'\mathbf{r} = r \\ (\mathbf{w}-\mathbf{b})'\boldsymbol{\Omega}(\mathbf{w}-\mathbf{b}) = T^2 \end{gathered} \tag{30}$$

Solving this system for **w** yields the following weight vector:

$$\mathbf{w} = \mathbf{b} + \left(\frac{\lambda_1\mathbf{1} + \lambda_r\mathbf{r} - 2\lambda_{T^2}\boldsymbol{\Omega}\mathbf{b}}{2(1-\lambda_{T^2})}\right) \tag{31}$$

And applying this weight vector to the variance calculation yields the following ellipse in mean-variance space. Jorion calls this the constant TEV frontier.

$$
\begin{bmatrix} \sigma^2 - \sigma_b^2 - T^2 \\ r - r_b \end{bmatrix}' \begin{bmatrix} \mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{r} - \frac{(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1})^2}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}} & -2\left(r_b - \frac{\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right) \\ -2\left(r_b - \frac{\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right) & 4\left(\sigma_b^2 - \frac{1}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right) \end{bmatrix} \begin{bmatrix} \sigma^2 - \sigma_b^2 - T^2 \\ r - r_b \end{bmatrix}
$$

$$
- T^2 \begin{vmatrix} \mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{r} - \frac{(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1})^2}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}} & -2\left(r_b - \frac{\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right) \\ -2\left(r_b - \frac{\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right) & 4\left(\sigma_b^2 - \frac{1}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right) \end{vmatrix} = 0 \tag{32}
$$

The notation |…| is the determinant of the matrix. This ellipse is illustrated in Figure 3, Panel A. Note that this is an ellipse in mean and variance and this figure is in mean and standard deviation, so the ellipse is slightly distorted. This ellipse grows and shrinks as the constant, T^2, is increased and decreased. However, since this is an ellipse, it is only defined over a limited region in *r*. As Jorion (2003) shows, this ellipse reaches its maximum and minimum values in return along the TEV frontier at the following levels, given a fixed *T*:

$$
r = r_b \pm T\sqrt{\left(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{r} - \frac{(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1})^2}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right)} \tag{33}
$$

This expression is actually equivalent to equation (13) from Section II of this essay, the TEV frontier in mean and tracking error volatility. Equation (13) is just solved for the return. The constant TEV frontier is graphed with the TEV frontier in relative space in Figure 3, Panel B. The constant TEV frontier is just a vertical line connecting the top of the curve to bottom of the curve and it should be evident from this figure that the maximum and minimum must be reached along the frontier.

(Insert Figure 3)

The expression with the radical from (33) is the slope of the line extending from the intercept and is also the maximum possible information ratio given the return vector and the covariance matrix. This is an important constant and it even appeared in equation (20) from this essay. Many other authors, stemming from Merton (1972), define this constant by giving it a name (usually *d*). The equivalence between constraining in tracking error volatility and agent risk aversion is also evident in equation (33)'s similarity to equation (19), the external manager's optimal choice given a level of risk aversion. If we take the transpose of equation (19) and multiply it by the return vector, we get the following:

$$\mathbf{w}'\mathbf{r} = \mathbf{b}'\mathbf{r} + \frac{1}{2\theta}\left(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{r} - \frac{\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{r}\right) \tag{34}$$

$$r = r_b + \frac{1}{2\theta}\left(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{r} - \frac{(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1})^2}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right) \tag{35}$$

And, by setting equation (35) equal to equation (33), equivalence between the agent's quadratic utility risk aversion coefficient, θ, and the tracking error volatility, T, can be obtained:

$$T = \frac{1}{2\theta}\sqrt{\left(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{r} - \frac{(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1})^2}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right)} \tag{36}$$

This shows that constraining on tracking error is a practical way of constraining delegated managers to act as if their risk aversion had been constrained. The constraint does not change the agent's level of risk aversion. It merely forces the external manager to buy a portfolio as if he had a higher level of risk aversion.

What is obvious when considering the utility maximization problem of the agent, equation (16), is that if tracking error is constrained to be fixed, then the agent's utility maximization problem just becomes a return maximization problem: maximize r given a fixed T.

$$\max_{\mathbf{w}} U = \max_{\mathbf{w}} \mathbf{w}'\mathbf{r} - \theta(\mathbf{w}-\mathbf{b})'\boldsymbol{\Omega}(\mathbf{w}-\mathbf{b}) \ , \tag{37}$$

$$s.t. \ \mathbf{w}'\mathbf{1} = 1 \ and \ (\mathbf{w}-\mathbf{b})'\boldsymbol{\Omega}(\mathbf{w}-\mathbf{b}) = T^2$$

Solving this problem will yield the following weight vector but equation (36) could also be substituted into equation (19) to arrive at the same result:

$$\mathbf{w} = T\left(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{r} - \frac{(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1})^2}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right)^{-1/2} \boldsymbol{\Omega}^{-1}\left(\mathbf{r} - \frac{\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\mathbf{1}\right) + \mathbf{b} \tag{38}$$

Note that this equation is independent of the agent's risk aversion parameter. Equivalently, the minimization problem for (37) could be solved and it yields a similar solution to (38) except that the first term is negative.

The return for this weight vector is already given in equation (35) and the tracking error is fixed. This could be verified by plugging the weight vector in (38) into the tracking error equation (11). The combination of this return and tracking error are plotted in Figure 3, Panels A and B. Additionally, the utility curve associated with this portfolio is plotted alongside the agent's maximal utility curve. This constraint is necessarily utility decreasing for the agent. The opposite is true for the principal however. Since the principal cares mostly about the variance of this weight vector (38), plugging this into the variance equation yields the following:

$$\mathbf{w}'\boldsymbol{\Omega}\mathbf{w} = T^2 + \sigma_b^2 + 2T\left(r_b - \frac{\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right)\left(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{r} - \frac{(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1})^2}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right)^{-1/2} \tag{39}$$

This is the variance of the portfolio above the benchmark on the TEV frontier as a function of tracking error, *T*. Similarly, the portfolio below the benchmark on the TEV frontier at the same level of tracking error has a variance with a similar equation but with one small difference: the third term in (39) is negative.

Given that the delegated manager will choose a portfolio on the TEV frontier, a principal can maximize his own utility relative to the TEV frontier by choosing the appropriate tracking error bound to apply to an external manager. Since the return and standard deviation can be parameterized in *T*, then we simply maximize the principal's utility function (21) with respect to *T* by substituting in equations (33) and (39) for the return and variance.

$$
\begin{aligned}
\max_T U = &\ \max_T \ \mathbf{w}'\mathbf{r} - \theta\mathbf{w}'\boldsymbol{\Omega}\mathbf{w} \\
&= \max_T \ r_b + T\left(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{r} - \frac{(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1})^2}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right)^{1/2} \\
&- \theta\left(T^2 + \sigma_b^2 \right. \\
&\left. + 2T\left(r_b - \frac{\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right)\left(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{r} - \frac{(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1})^2}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right)^{-1/2}\right)
\end{aligned}
\tag{40}
$$

And the solution is:

$$
\begin{aligned}
T = &\ \frac{1}{2\theta}\left(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{r} - \frac{(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1})^2}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right)^{1/2} \\
&- \left(r_b - \frac{\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right)\left(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{r} - \frac{(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1})^2}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right)^{-1/2}
\end{aligned}
\tag{41}
$$

This is the tracking error bound a principal with quadratic utility should choose to force the delegated manager to buy a portfolio that maximizes the principal's utility subject to the constraint of the TEV frontier. If this bound is chosen, and the agent abides by the constraint, then as is depicted in Figure 3, Panel A, the principal's utility curve associated with this constraint is tangent to the TEV frontier at the intersection of the frontier with the constant TEV ellipse. This level of utility is preferred to the one associated with holding the benchmark albeit still less preferred than the one associated with the principal's global optimization problem.

Table 2 calculates the quadratic utility deviations for the principal under differing scenarios of agent and principal risk aversion. Panel A depicts the utility increase or decrease for the principal from a fixed tracking error constraint of 8%. The delegated performance incentive with a tracking error constraint has the potential to increase utility for the principal if the constraint is chosen appropriately. Given the levels of utility I use in the figures for this paper, the utility increases by 0.20% through delegation with an 8% constraint. Panel B shows the utility increase from the benchmark given an optimal tracking error constraint; this is necessarily non-negative. Also depicted is that the level of utility increase is independent of agent risk aversion. Essentially, values down each column is identical. Recall that the tracking error constraint substitutes for the agent's risk aversion level.

(Insert Table II)

V. Constraining Benchmark Relative Beta

In addition to the tracking error constraint, another pervasive constraint applied in the modern investment industry is the benchmark relative beta constraint. The colloquial way to consider a beta constraint is to put a limitation on "style drift" but the beta constraint is about much more than just the technical definition of style drift. Most delegated investment management is done in two stages: a strategic allocation stage, where the asset class weights (the beta or factor sensitivities) are set, then the tactical asset allocation is made to improve performance over each factor. When hiring an external manager to manage all or part of an allocation to a particular asset class, the principal is mostly concerned with whether the agent manages the factor sensitivity appropriately. If an agent creates factor sensitivity different than β = 1 against the benchmark, this hurts the optimality of the strategic allocation either by altering the agent's mandated factor sensitivity and therefore the effective weight of that allocation in the overall portfolio, or though the agent cannibalizing factor sensitivity from a different delegated manager. However, even in the case of a single delegated manager, the beta constraint seems justified. Agents are being hired to use their skill to the benefit of the principal. As per our assumptions, the principal already has the skill to buy and therefore lever the factor sensitivity of the benchmark, at least downward. Thus some form of beta constraint seems rational when trying to control delegated managers.

Many external managers will argue that the beta constraint limits their ability to generate returns for the principal. And, just like the tracking error constraint, the beta constraint necessarily reduces the utility of the agent but has the potential to increase utility for the principal. Roll (1992) analyzes the beta constraint and shows that when the benchmark is above the minimum variance portfolio in return the beta constrained TEV frontier is superior to the unconstrained TEV frontier for some finite region above the benchmark portfolio. We can set up the minimization problem as follows.

$$\min_{\mathbf{w}}(\mathbf{w}-\mathbf{b})'\boldsymbol{\Omega}(\mathbf{w}-\mathbf{b}) \quad s.t.\ \mathbf{w}'\mathbf{1}=1\ \mathbf{w}'\mathbf{r}=r\ and\ \mathbf{w}'\boldsymbol{\Omega}\mathbf{b}=\beta\sigma_b^2 \qquad (42)$$

This problem is actually set up with a constraint to covariance, $\mathbf{w}'\boldsymbol{\Omega}\mathbf{b}$, rather than beta but since the variance of the relative benchmark is a constant, constraining on beta and constraining on covariance are equivalent. Also, it should be noted that under the constraint on beta, or covariance, the minimization problem on tracking error and the minimization problem on variance are also equivalent since the tracking error is now a function of the variance of the portfolio and two constants, covariance and the variance of the benchmark. Thus the following problem could be solved for **w** and an identical solution would be obtained.

$$\min_{\mathbf{w}} \mathbf{w}'\boldsymbol{\Omega}\mathbf{w} \quad s.t.\ \mathbf{w}'\mathbf{1} = 1\ \mathbf{w}'\mathbf{r} = r\ and\ \mathbf{w}'\boldsymbol{\Omega}\mathbf{b} = \beta\sigma_b^2 \tag{43}$$

Unlike the situation with unconstrained TEV optimization, under the condition of a beta constraint, the delegated manager and the principal are, at least, now optimizing against the same curve. Below is the Lagrangian based on equation (43)

$$\begin{aligned} &\mathrm{L}(\mathbf{w}, \lambda_1, \lambda_r, \lambda_\beta) \\ &= \mathbf{w}'\boldsymbol{\Omega}\mathbf{w} - \lambda_1(\mathbf{w}'\mathbf{1} - 1) - \lambda_r(\mathbf{w}'\mathbf{r} - r) - \lambda_\beta(\mathbf{w}'\boldsymbol{\Omega}\mathbf{b} - \beta\sigma_b^2) \end{aligned} \tag{44}$$

Differentiating yields the following system of equations:

$$\begin{gathered} 2\boldsymbol{\Omega}\mathbf{w} - \lambda_1\mathbf{1} - \lambda_r\mathbf{r} - \lambda_\beta\boldsymbol{\Omega}\mathbf{b} = \mathbf{0} \\ \mathbf{w}'\mathbf{1} = 1 \\ \mathbf{w}'\mathbf{r} = r \\ \mathbf{w}'\boldsymbol{\Omega}\mathbf{b} = \beta\sigma_b^2 \end{gathered} \tag{45}$$

Solving this system yields the following weight vector:

$$\mathbf{w} = \boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r} \quad \boldsymbol{\Omega}\mathbf{b}]\left[[\mathbf{1} \quad \mathbf{r} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r} \quad \boldsymbol{\Omega}\mathbf{b}]\right]^{-1}\begin{bmatrix} 1 \\ r \\ \beta\sigma_b^2 \end{bmatrix} \tag{46}$$

And the equation for the parabola in mean-variance space is derived by plugging the weight vector from (46) into the variance equation as follows. This curve is depicted in Figure 4, Panel A for the cases $\beta = 1$, $\beta > 1$, and $\beta < 1$.

$$\sigma^2 = \mathbf{w}'\boldsymbol{\Omega}\mathbf{w} = \begin{bmatrix} 1 \\ r \\ \beta\sigma_b^2 \end{bmatrix}' \left[[\mathbf{1} \quad \mathbf{r} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r} \quad \boldsymbol{\Omega}\mathbf{b}]\right]^{-1} \begin{bmatrix} 1 \\ r \\ \beta\sigma_b^2 \end{bmatrix} \tag{47}$$

The equation for the curve in relative mean-tracking error space can be computed by first differencing the weight vector in (46) with $\mathbf{b}$:

$$(\mathbf{w} - \mathbf{b}) = \boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r} \quad \boldsymbol{\Omega}\mathbf{b}] \left[[\mathbf{1} \quad \mathbf{r} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r} \quad \boldsymbol{\Omega}\mathbf{b}]\right]^{-1} \begin{bmatrix} 1 \\ r \\ \beta\sigma_b^2 \end{bmatrix} - \mathbf{b} \tag{48}$$

Next, the differenced vector in equation (48) can be plugged into the tracking error variance equation to yield the following:

$$\begin{aligned} T^2 &= \begin{bmatrix} 1 \\ r \\ \beta\sigma_b^2 \end{bmatrix}' \left[[\mathbf{1} \quad \mathbf{r} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r} \quad \boldsymbol{\Omega}\mathbf{b}]\right]^{-1} \begin{bmatrix} 1 \\ r \\ \beta\sigma_b^2 \end{bmatrix} + \sigma_b^2 \\ &\quad - 2\begin{bmatrix} 1 \\ r_b \\ \sigma_b^2 \end{bmatrix}' \left[[\mathbf{1} \quad \mathbf{r} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r} \quad \boldsymbol{\Omega}\mathbf{b}]\right]^{-1} \begin{bmatrix} 1 \\ r \\ \beta\sigma_b^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ r \\ \beta\sigma_b^2 \end{bmatrix}' \left[[\mathbf{1} \quad \mathbf{r} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \mathbf{r} \quad \boldsymbol{\Omega}\mathbf{b}]\right]^{-1} \begin{bmatrix} 1 \\ r \\ \beta\sigma_b^2 \end{bmatrix} \\ &\quad + (1 - 2\beta)\sigma_b^2 \end{aligned} \tag{49}$$

This curve is depicted in Figure 4, Panel B also for three cases, $\beta = 1$, $\beta > 1$, and $\beta < 1$. Note that these curves are all at best as good at the TEV frontier in preference and a principal requiring a beta constraint will most likely require the agent to suffer a utility deterioration. The weight vector the external manager will choose can be derived by maximizing his utility function however this time with the addition of a beta constraint.

Just as with the derivation of the beta constrained TEV frontier, maximizing the agent's utility function parameterized in tracking error and maximizing the principal's utility function parameterized in variance yields the same choice vector, given the same value for θ, because the beta, or covariance, constraint makes $\mathbf{w}'\boldsymbol{\Omega}\mathbf{w}$ the only free variable in the risk calculation. However, there is no reason why the risk aversion parameter would be the same so it is still likely an agent and a principal would choose a different portfolio. Below is the utility to be optimized, parameterized in variance.

$$\max_{\mathbf{w}} U = \max_{\mathbf{w}} \mathbf{w}'\mathbf{r} - \theta\mathbf{w}'\boldsymbol{\Omega}\mathbf{w} \quad s.t.\ \mathbf{w}'\mathbf{1} = 1\ and\ \mathbf{w}'\boldsymbol{\Omega}\mathbf{b} = \beta\sigma_b^2 \tag{50}$$

$$\mathrm{L}(\mathbf{w}, \lambda_1, \lambda_\beta) = \mathbf{w}'\mathbf{r} - \theta\mathbf{w}'\boldsymbol{\Omega}\mathbf{w} - \lambda_1(\mathbf{w}'\mathbf{1} - 1) - \lambda_\beta(\mathbf{w}'\boldsymbol{\Omega}\mathbf{b} - \beta\sigma_b^2) \tag{51}$$

Differentiating the Lagrangian, finding the critical value and solving for **w** yields the following vector:

$$\mathbf{w} = \frac{1}{2\theta}\boldsymbol{\Omega}^{-1}\left[\mathbf{r} - [\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]\left[[\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]\right]^{-1}\left[[\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}\mathbf{r} - 2\theta\begin{bmatrix}1 \\ \beta\sigma_b^2\end{bmatrix}\right]\right] \tag{52}$$

This is the weight vector for the portfolio that the principal and the agent would choose given a risk aversion level of θ and a beta constraint of β.

The idea of the principal choosing an optimal beta constraint is slightly more ambiguous than the principal's choice of a tracking error constraint. The tracking error problem eliminated the need to estimate the agent's risk aversion, but the risk aversion level is still a prominent parameter in the beta constrained TEV frontier as is evident in equation (47). The delegated manager's weight vector in (52) could be used in the portfolio variance and return functions, plugged into the principal's utility function, and that function could be solved for the critical value on beta to arrive at the optimal beta. But, there is an easier way to discover this value by inspection. Recognize that since the optimization happens on the same curve in absolute and relative space, if the principal and agent have the same value for their individual risk aversion constants, the principal could constrain the agent based on the beta of the global optimal portfolio and the agent would buy the global optimal portfolio. However, it is more likely that their risk aversion levels differ. But, this is not a concern. When you plug the weight vector from (52) into the principal's utility function and differentiate on beta, the agent's risk aversion constants cancel in the only remaining terms. Thus, the optimal beta constraint is independent of the agent's risk aversion level. And, since we already discovered it given equal risk aversion levels, it must be equal to this value, the beta of the global optimal portfolio. Therefore, to obtain the optimal beta constraint, all one needs is to go back to the portfolio choice made in the principal's utility given the principal's risk aversion level of θ, and calculate the beta of the optimal choice portfolio. Beta is calculated as follows:

$$\beta = \frac{1}{\sigma_b^2}\mathbf{w}'\mathbf{\Omega}\mathbf{b} \tag{53}$$

Thus, we simply plug-in the weight vector from equation (24) and the optimal beta is calculated:

$$\beta = \frac{1}{2\theta\sigma_b^2}\left(\mathbf{r}'\mathbf{b} - \frac{\mathbf{r}'\mathbf{\Omega}^{-1}\mathbf{1} - 2\theta}{\mathbf{1}'\mathbf{\Omega}^{-1}\mathbf{1}}\mathbf{1}'\mathbf{b}\right) = \frac{1}{2\theta\sigma_b^2}\left(r_b - \frac{\mathbf{r}'\mathbf{\Omega}^{-1}\mathbf{1} - 2\theta}{\mathbf{1}'\mathbf{\Omega}^{-1}\mathbf{1}}\right) \tag{54}$$

This result could be verified by going through the standard process of maximizing a utility function described in the last paragraph. This optimal beta constrained TEV curve is depicted in Figure 4, Panels A and B and in this case is represented by the curve where $\beta < 1$. As is evident from the figure, the iso-utility curves are plotted for an agent with a higher risk appetite than the principal. The agent's utility curve under a beta constraint is always below his optimal utility curve and therefore this is a utility decrease for the agent. However, as depicted in the figure, this constraint could be a possible utility increase for the principal.

(Insert Figure 4)

Table 3 calculates the quadratic utility deviations for the principal under differing scenarios. Panel A depicts the utility increase or decrease for the principal from a fixed Beta constraint of 1. The delegated performance incentive with a beta constraint has the potential to increase utility for the principal if the constraint is chosen appropriately. Given the levels of utility I use in the figures for this paper, the utility increases by 1.90% through delegation and a beta of 1. Although utility increase is relatively certain, if the agent's risk aversion level is low enough, he can still destroy utility under a beta constraint. Panel B shows the utility increase from the benchmark given an optimal beta constraint; this is necessarily better than Panel A. Essentially, cell my cell the utility level is higher in Panel B than in Panel A. Note that just like Panel A, the level of utility increase is substantial and relatively certain under reasonable levels of risk aversion, but still not guaranteed.

(Insert Table III)

VI. Maximizing Principal Utility by Constraining TEV and Beta

In section IV, I discussed controlling the level of risk an external manager takes by using a tracking error constraint, similar to Jorion (2003). This constrains the delegated manager to purchase a portfolio that optimizes principal utility despite the agent's actual level of risk aversion. In section V, I discussed using a beta constraint to encourage the external manager to invest on a frontier superior (at least for some finite region) to the TEV frontier, similar to Roll (1992). Both of these constraints have the ability to increase the principal's utility and have the unfortunate downside of decreasing agent utility. The combination of these two constraints should allow the principal to potentially increase utility to a greater extent than either constraint can accomplish alone. The inclusion of both a beta and tracking error constraint limits the set of possible portfolios to either 2, 1, or 0 real solutions in both mean-variance and mean-TEV space. The evidence that there are at most 2 solutions to these intersections is more technically understood from an analysis of equation (49). This is an equation parameterized in T, β, and r. T and β are constrained to be fixed, and since the equation is quadratic in r, it has a maximum of 2 solutions.

Since the values for T and β are derived from the same data, there are combinations of these calculations that are not feasible in a real sense based on the return vector and the covariance matrix, thus the option of zero solutions is a definite possibility. Also as was discussed in the analysis of the agent's portfolio choice along the TEV frontier, if there are two solutions to the optimization problem, one of the solutions always dominates the other solution in the agent's portfolio choice based on his utility function. Thus the proper use of a beta and TEV constraint, as long as the combination is consistent, could direct the external manager to buy any portfolio underneath the envelope of the efficient frontier. "Any portfolio" obviously includes the principal's global optimal portfolio. Therefore, a principal would calculate the optimal constraints by maximizing his global utility function, unconstrained, then merely calculate the beta and tracking error of this portfolio and provide those constraints to the agent. Again, this would force the agent to buy the principal's global optimal portfolio.

The optimal beta constraint was calculated in the previous section and recall that this constraint is not dependent on the risk aversion level of the agent. Thus, the optimal beta constraint is always as is given in equation (54). Additionally, the tracking error constraint requirement made the agent's risk aversion irrelevant. However, just as with controlling agent risk level along the TEV frontier by increasing and decreasing tracking error, the principal controls the agent's risk level along the beta constrained frontier with a tracking error constraint. As the tracking error constraint is increased, the agent moves further and further up the constrained beta frontier until the point where the constraint is reached, at the optimal portfolio. The tracking error variance of the global optimal portfolio can be calculated by using the weight vector of the principal's optimal portfolio choice from equation (24) and substituting it into the definition of tracking error variance:

$$T^2 = \frac{1}{2\theta}\left(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{r} - 2\left(\frac{\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1} - 2\theta}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right)\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1} + \frac{(\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1} - 2\theta)^2}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right) + \sigma_b^2 - \frac{1}{\theta}\left(r_b - \frac{\mathbf{r}'\boldsymbol{\Omega}^{-1}\mathbf{1} - 2\theta}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}}\right) \tag{55}$$

This is the tracking error of the principal's global optimal portfolio given an optimization on the agent's information. In this context, the tracking error constraint is definitely a function of the principal's risk aversion coefficient.

In Figure 5, Panel A and B the curves of the mean-variance frontier, the constant TEV ellipse, and the constant beta frontier are graphed for various scenarios. Additionally, the global maximum portfolio is identified and the associated iso-utility curves are drawn; the principal's utility is depicted in Panel A and the agent's utility is depicted in Panel B. It is evident in the figure that the constant beta frontier can cross the constant TEV ellipse in at most 2 places and also that it is possible for the curves to have no real intersections, indicating a set of inconsistent constraints. The constant beta frontier is also tangent to the mean-variance frontier at exactly one point. If the tracking error and beta constraints are as was defined above, then the curves of the efficient frontier, the constant beta frontier, and the constant TEV ellipse are exactly tangent at one point, the global mean-variance optimal portfolio. Thus the intersection of these curves reflects that the constraints incentivize the delegated manager to buy the principal's global optimal portfolio.

(Insert Figure 5)

However, the constraints associated with the global maximum portfolio may be too loose. For example, in the dataset used to build the figures in this paper, the beta constraint of the global optimal portfolio is about 0.68 and the tracking error volatility constraint is about 10%. These constraints seem rather ridiculous to give an external manager in the context of delegated management. In particular, the concern with a loose tracking error constraint is whether it is even possible to generate that type of tracking error versus a benchmark without buying another uncorrelated systematic risk. This theory implies that all of the differentiation is done with the delegated manager's uncorrelated alpha generating process. Which leads to the second concern: if you benchmark a manager and give him a beta constraint of 0.6, what does the manager do with the other 0.4? If the external manager's uncorrelated, idiosyncratic strategy, their alpha strategy, takes factor exposure to any asset class other than the mandated asset class, then that creates inefficiency in in the principal's overall portfolio.

That however is only the first concern. It is not only probable but highly likely that it is not possible for the principal to identify his global optimal portfolio much less calculate the tracking error variance and beta of the portfolio. This is particularly true considering the set of information he would need to optimize over is the agent's set of information, information for which he would be hiring the delegated manager because the principal does not know this information. As other authors have suggested, using general bounds for these two constraints is probably more realistic. Roll (1992) emphasizes the beta constraint of $\beta = 1$. This constraint has the benefit of a guaranteed utility increase no matter the slope of the principal's utility and is also consistent with the idea of preventing style drift. The tracking error constraint can be imposed to move up the beta constrained frontier. Below I calculate the optimal tracking error constraint given a beta constraint of 1; however, it is likely that a moderate tracking error constraint would also be good enough. Not coincidently, these two constraints, with reasonable values, are exactly the way principals control delegated managers in the investment industry.

Thus, the problem is to optimize principal utility by choosing tracking error given the constraint of $\beta = 1$. We could parameterize the principal's utility as a function of tracking error. Similar to the problem of optimizing agent utility given a tracking error constraint, under the conditions of a fixed tracking error and a fixed beta, optimizing either agent or principal utility is now just a return maximization problem. The relationship between tracking error and variance is a simple one, particularly with $\beta = 1$. The expression is just as follows:

$$\sigma^2 = T^2 + \sigma_b^2 \tag{56}$$

Calculating the level of return associated with the two constraints is a rather complicated expression but could be done by solving equation (49) for r. It could also be done by using equation (52) above, setting beta equal to zero and using the principal's risk aversion instead of the agent's risk aversion.

$$\mathbf{w} = \frac{1}{2\theta}\boldsymbol{\Omega}^{-1}\left[\mathbf{r} - [\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]\big[[\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]\big]^{-1}\left[[\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}\mathbf{r} - 2\theta\begin{bmatrix}1\\ \sigma_b^2\end{bmatrix}\right]\right] \tag{57}$$

Plugging this weight vector into the return function yields the following:

$$r = \mathbf{w}'\mathbf{r} = \frac{1}{2\theta}\left[\mathbf{r} - [\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]\big[[\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]\big]^{-1}\left[[\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}\mathbf{r} - 2\theta\begin{bmatrix}1\\ \sigma_b^2\end{bmatrix}\right]\right]'\boldsymbol{\Omega}^{-1}\boldsymbol{r} \tag{58}$$

And the optimal constraint to T is chosen by applying that weight vector to the tracking error function:

$$T^2 = \frac{1}{(2\theta)^2}\left[\mathbf{r} - [\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]\big[[\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]\big]^{-1}\left[[\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}\mathbf{r} - 2\theta\begin{bmatrix}1\\ \sigma_b^2\end{bmatrix}\right]\right]'\boldsymbol{\Omega}^{-1}\left[\mathbf{r} - [\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]\big[[\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}[\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]\big]^{-1}\left[[\mathbf{1} \quad \boldsymbol{\Omega}\mathbf{b}]'\boldsymbol{\Omega}^{-1}\mathbf{r} - 2\theta\begin{bmatrix}1\\ \sigma_b^2\end{bmatrix}\right]\right] \tag{59}$$

This is the optimal constraint to *T* given a beta of 1 and the agent's optimization process in relative space.

Just for reference, the numerical value of this point in my figures is about 9%. Realistically, this may also be a particularly loose tracking error constraint. Even if a constraint is lower than where the optimal point is chosen, this is still guaranteed to be a utility increase for the principal since any deviation from the benchmark below the optimal point and above the benchmark must be a utility increase. Jorion (2003) agrees with the idea of a general tracking error constraint. His analysis brings up the point of using a tracking error bound to not extend variance too far and I am emphasizing that the bound should be set to maximize utility. But, those two goals consistent with each other once the optimal tracking error bound is reached.

The constraints described above are depicted in Figure 6, Panels A and B. It is evident from the figure that the constrained beta frontier when $\beta = 1$ passes thought the benchmark at its minimum point. The frontier is superior to the TEV frontier over a good portion of its region, and the curve is tangent to the efficient frontier at a point well up the curve from the global efficient portfolio. Increasing constant TEV ellipses eventually at the point of the optimally constrained portfolio are depicted. The principal's iso-utility curves are included in Panel A at the level of the benchmark, the optimally constrained portfolio, and the global optimal portfolio. Utility at the optimally constrained portfolio has increased from the benchmark and the utility curve is tangent to the constrained beta frontier at this point. Additionally, the optimal TEV ellipse passes through the chosen portfolio at this point. The utility of this portfolio is obviously less than the utility of the global optimal but recall that it is unlikely that the principal knows the location of this portfolio anyway. The agent's utility curves are drawn in Panel B along with the optimization curves and it is apparent that utility in this space decreases for the agent.

(Insert Figure 6)

VII Conclusion

This paper revisits the problem of Roll (1992) where a delegated investment manager optimizes over tracking error volatility rather than standard deviation. In comparison to mean-variance optimization, TEV optimization creates a frontier that is inferior to the efficient frontier. I show how this directly implies that the agent is optimizing over a utility function that is parameterized in tracking error volatility rather than in variance. This framework is a prominent feature of modern investment management and arises through both a direct and an indirect performance incentive. This performance incentive has been studied by many other authors and they arrive at various conclusions, mostly implying that relative optimization over tracking error is inefficient. Therefore delegated management, because of this inefficiency, is likely to be inferior for a principal even if the delegated manager has more skill and information.

However, I show that the principal's preference is really dependent upon the interaction of the TEV frontier and the principal's utility function. Except for a very unlikely case, the case where the benchmark is perfectly optimal over the agent's information, substantial utility improvements exist for the principal if the delegated agent is properly controlled. Given industry anecdotes related to tracking error constraints and beta constraints, I reanalyze the tracking error constraint from Jorion (2003) and the beta constraint from Roll (1992) in the context of a principal's utility function to show that there are levels of these constraints that likely increase utility for the principal.

In isolation, the tracking error constraint is used to limit the level of standard deviation an unconstrained delegated manager would likely take. Without the tracking error constraint, an agent could buy a portfolio so far up the TEV frontier that it actually decreases the principal's utility therefore making the delegation a bad idea. By imposing this constraint, the principal could ensure that the agent buy a portfolio within the region that increases utility. If "good enough" information is available to the principal, I show how the principal could optimize the level of the tracking error constraint (given quadratic utility) thus, by imposing this optimal TEV constraint, could maximize his utility given the delegated performance incentive. This level of utility is necessarily a utility increase except for the special case mentioned above. I also show how the TEV constraint is effectively just a risk aversion constraint. So, rather than trying to constrain or filter delegated managers directly on their level of risk aversion, tracking error allows principals to force managers to invest as if their risk aversion level was known to the principal. The agent's actual level of risk aversion is an unnecessary element of how to set the tracking error constraint. Only the principal's risk aversion level matters.

Additionally, as Roll (1992) shows, the beta constrained TEV frontier is superior to the unconstrained TEV frontier and therefore is also a guaranteed utility increase for the principal as long as the beta constraint is set appropriately. A beta constraint of 1 always has a positive utility deviation for the principal. Just as an optimal tracking error can be calculated, there is an optimal beta constraint that can be used to maximize the principal's utility. Perhaps surprisingly, this beta constraint is constant despite the level of risk aversion of agent and is, once again, only dependent on the level of the principal's risk aversion. It happens to be equal to the beta of the principal's global optimal portfolio, the portfolio optimized in principal utility over the agent's information. Choosing this beta constraint is necessarily a utility increase for the principal and thus makes a beta constrained portfolio, in the context of delegated portfolio management a good idea.

Given that these two constraints both work well in isolation, it is likely that a combination of these constraints lead to an even better portfolio. Additionally, because of the intersection of the beta constrained frontier with the constant TEV ellipse, and the agent's utility function, these two constraints can be used to pin a delegated manager to exactly one point underneath the envelope of the efficient frontier. Since this includes the principal's global optimal portfolio, setting the tracking error constraint and the beta constraint, which is just the same as the optimal beta constraint in isolation, to the levels of the global optimal portfolio forces the delegated manager to buy the optimal portfolio. However, since it is unlikely that a principal has the information to make this calculation, setting the constraints to reasonable levels are also likely to generate a utility increase for the principal. Following Roll and Jorion, who both suggest considering reasonable levels for these constraints rather than a rigorous calculation, I assume that the beta constraint is set to one, the case with an unambiguous utility increase. Under this assumption, I calculate the optimal tracking error constraint but it should be noted that any reasonable tracking error constraint under this scenario is almost certainly a utility increase for the principal.

Overall, delegated portfolio management is less efficient that doing it yourself. However, the reality of asymmetric information and differences in skill necessitate the delegation of a good portion of the assets that are managed in the modern investment industry. Even given the inefficiency of delegated management, it is still possible to create utility increases for the principal if constraints are properly imposed. In fact, under the scenario built in this essay, it is even possible to constrain the agent to act in a manner identical to that which the principal would act given exactly the same information and skill level. This work is supportive of the idea of delegated portfolio management and shows clearly that it can be an efficient and rational exercise if the delegated managers are properly controlled and not left to maximize their own (agent) utility at the expense of the principal.

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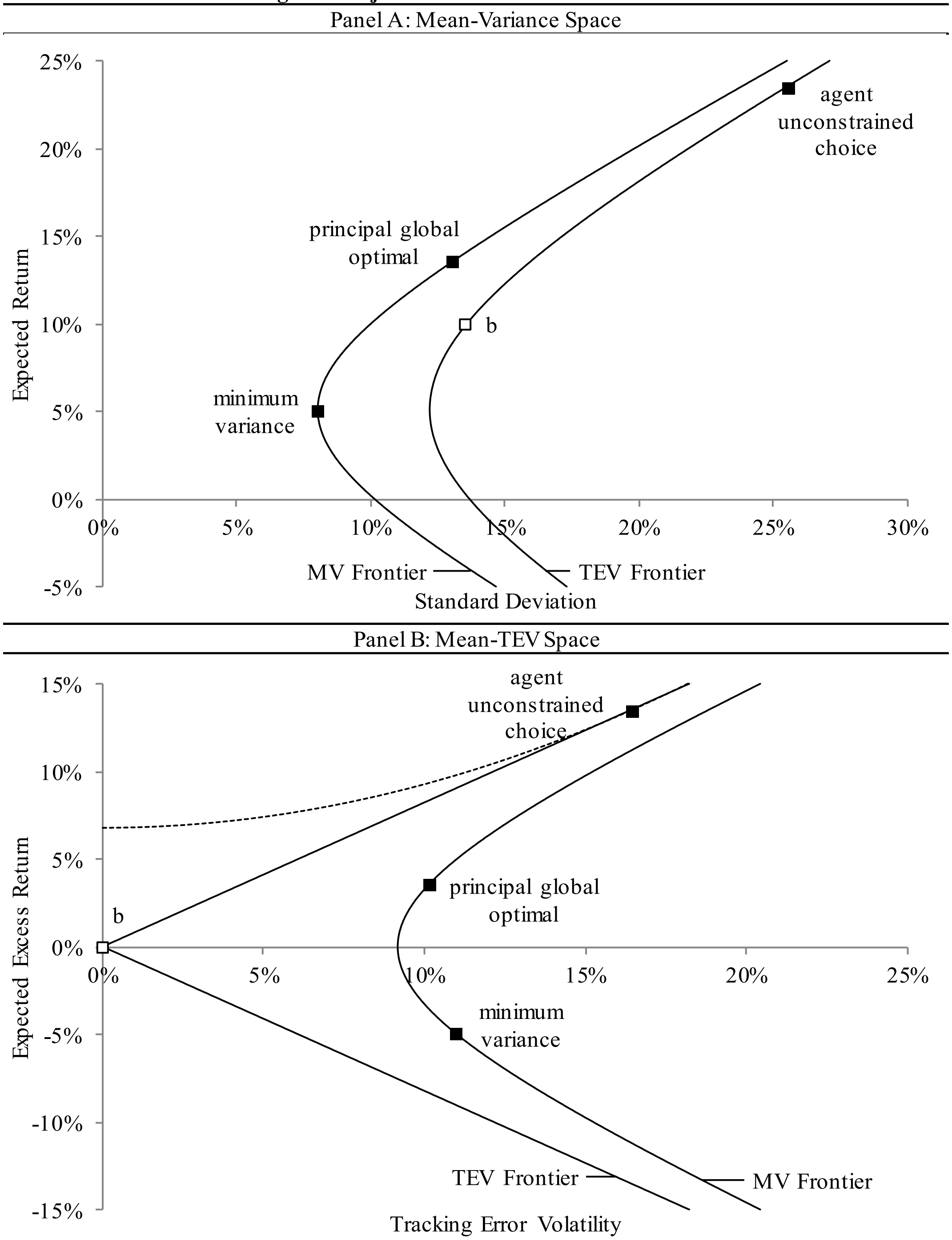
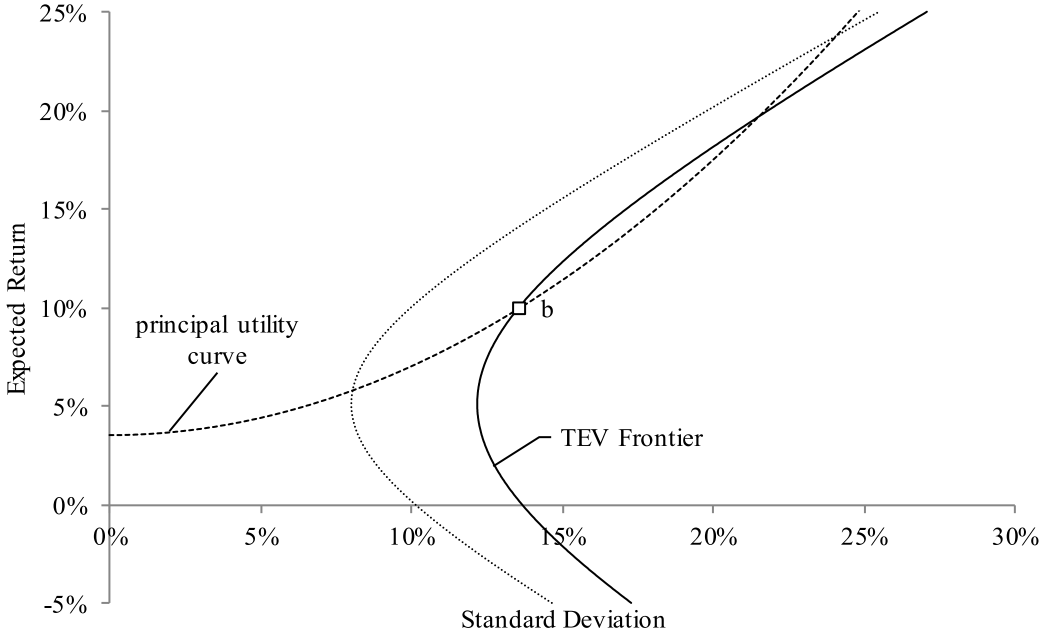
Figure 1
The TEV Frontier and the Agent's Objective Function



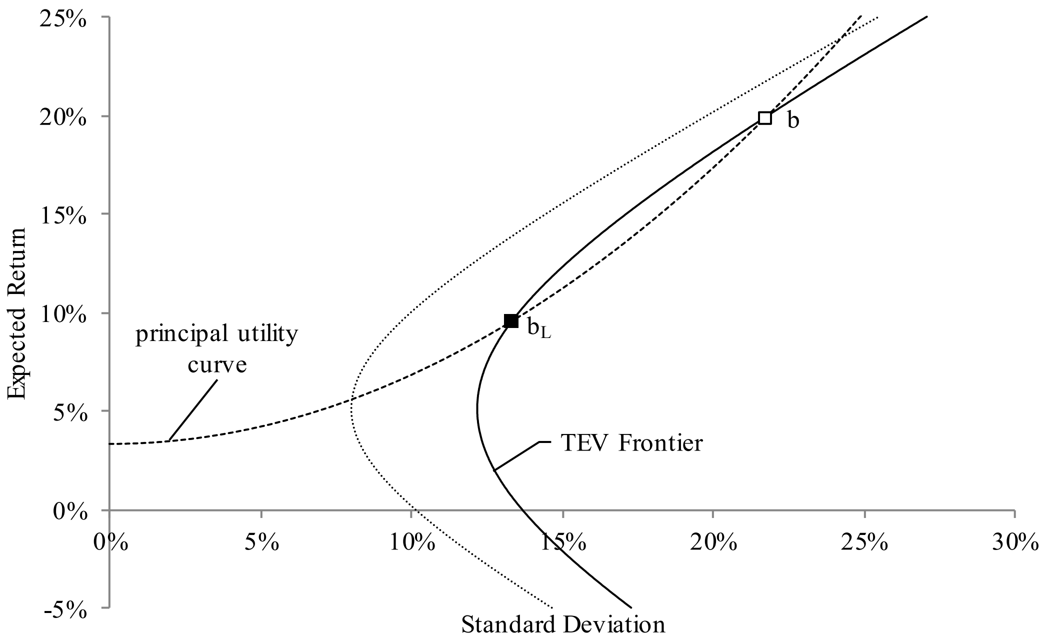
This figure shows how an agent, when maximizing utility in mean-TEV space, chooses a portfolio along the TEV frontier in Panel B. This portfolio is translated back into mean-variance space, in Panel A, and as is evident, the portfolio lies along the sub-optimal TEV frontier in that space as well.

Figure 2
Different Scenarios of how the TEV Frontier can Interact with Principal Utility

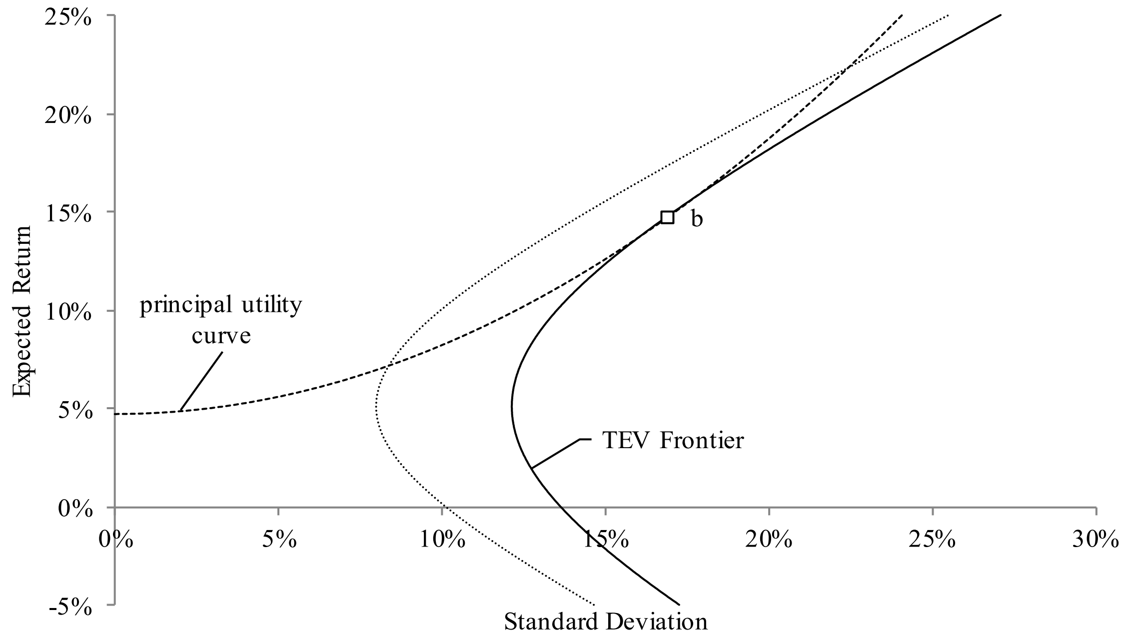
Panel A: TEV Frontier Steeper than Utility Curve at b



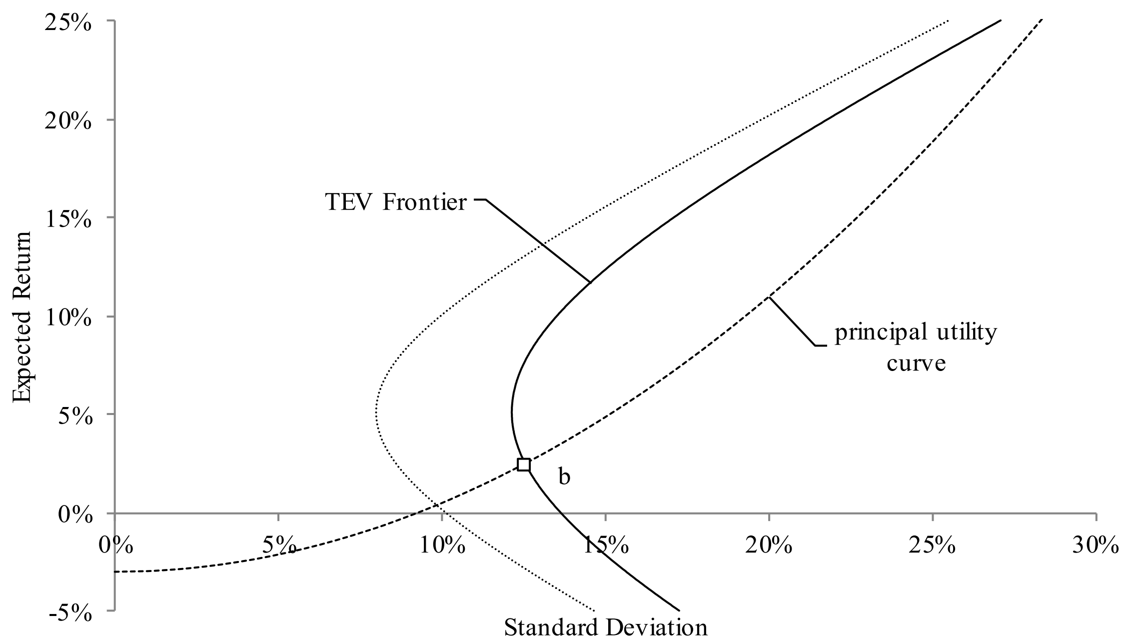
Panel B: TEV Frontier Flatter than Utility Curve at b



Panel C: TEV Frontier Steeper tangent to Utility Curve at b



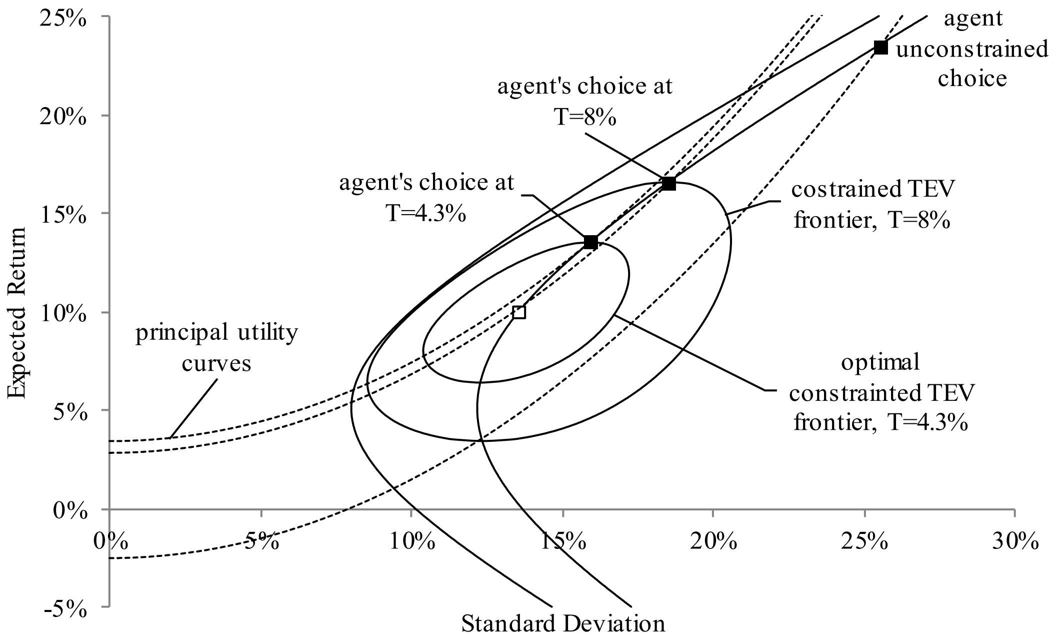
Panel D: Benchmark portfolio, b, Below the Minimum Variance Portfolio



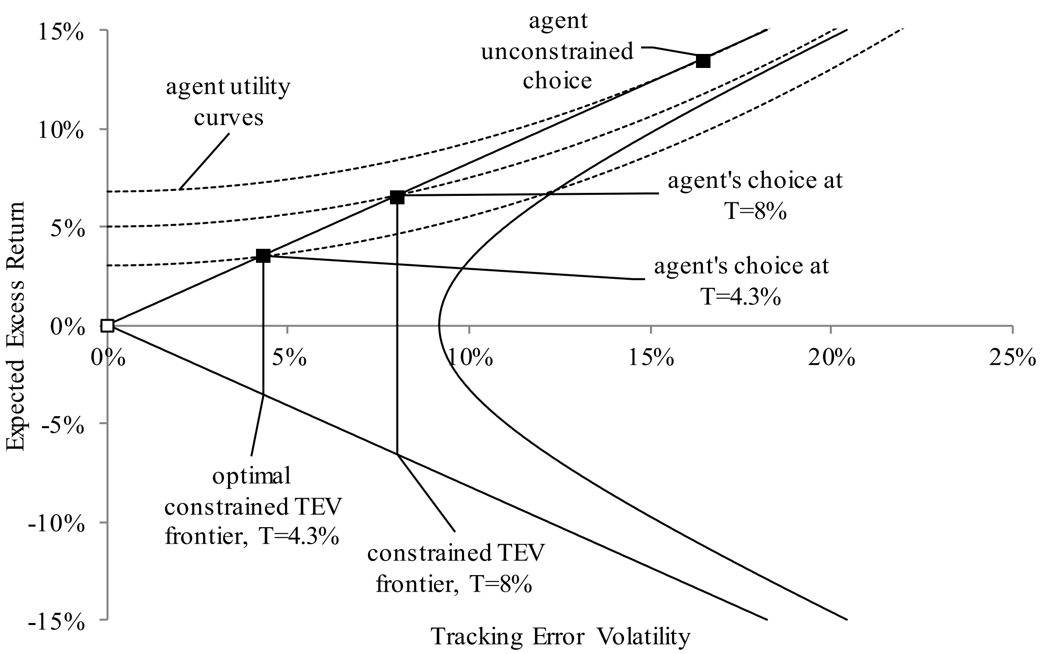
This figure depicts the 4 cases of interaction between the TEV frontier and the principal's iso-utility curve. The utility curve of the principal passes through the benchmark, b, and this is the point the principal is trying to improve upon through delegation. The MV frontier is shown as a lightly dotted line. Except for the case of Panel C, there is always a utility improvement for the principal along the TEV frontier.

Figure 3
Constraining the Agent's Risk Appetite using a Tracking Error Constraint

Panel A: Mean-Variance Space

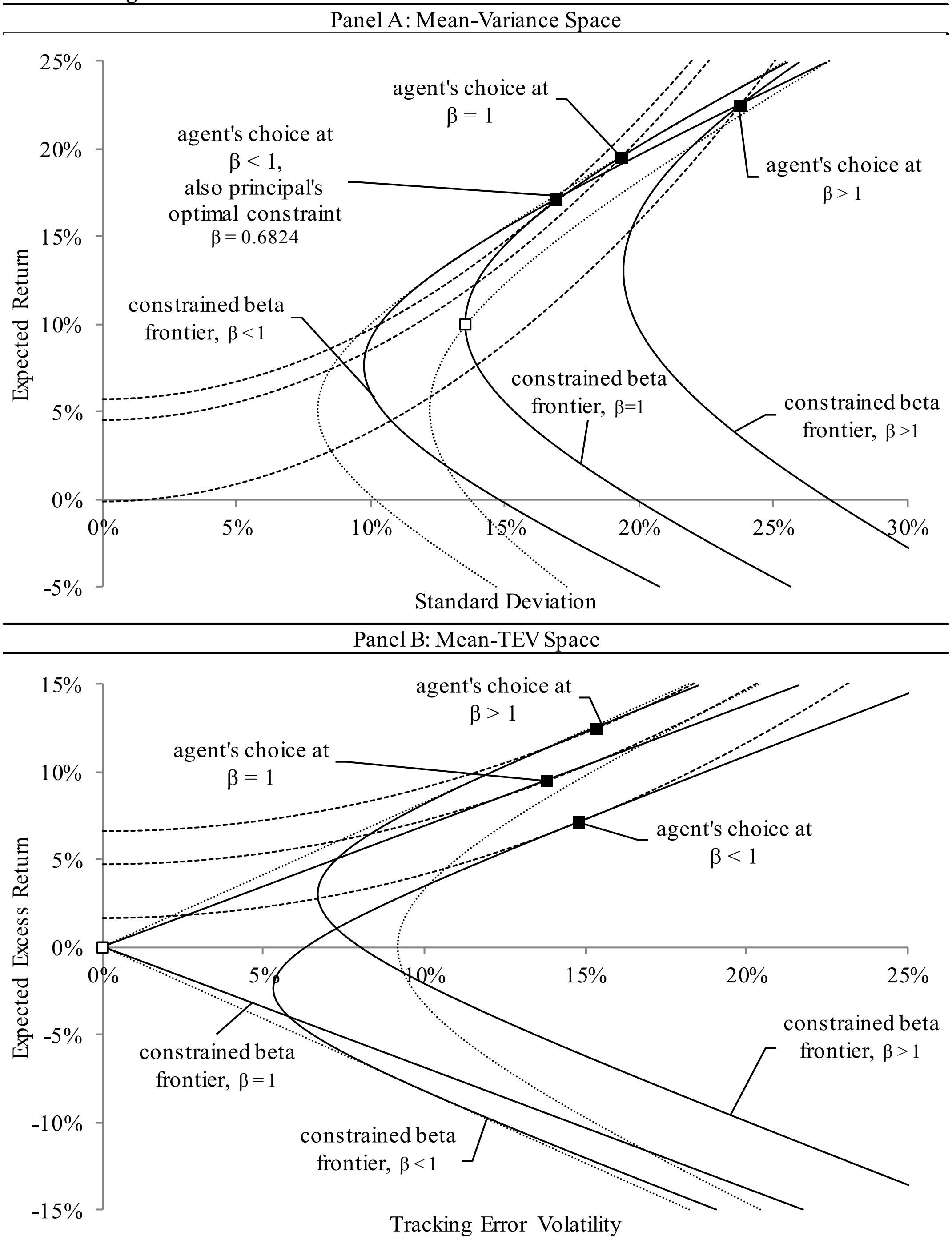


Panel B: Mean-TEV Space



This figure shows that constraining an agent on tracking error causes the agent to chose a portfolio further down the TEV frontier. This decreases agent utility but has the potential to increase principal utility. Depicted in this figure are an arbitrary constraint of T=8% and the principal's optimal constraint of about 4.3%. Note that the principal's utility curve is tangent to the TEV frontier at the optimal tracking error constraint.

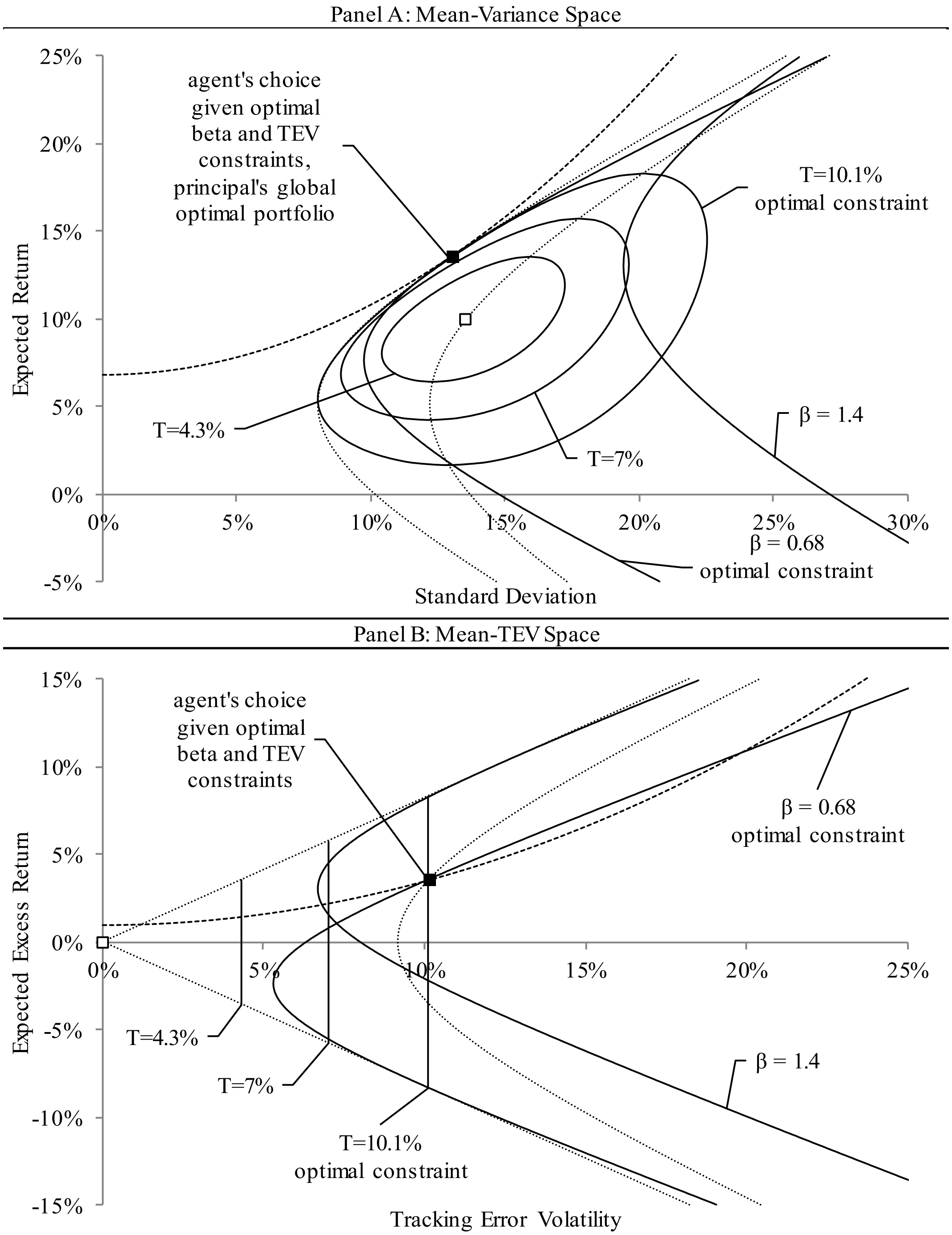
Figure 4
Constraining Benchmark Relative Beta



This figure demonstrates how the agent's portfolio choice changes with the benchmark relative beta constraint. In general, the agent's utility is increasing in beta. However, there is an optimal beta constraint for the principal. This constraint is depicted as the β < 1 curve. Not coincidently, this is the beta of the principal's global optimal portfolio.

Figure 5

Maximizing Principal Utility by Constraining Both the Tracking Error and Beta

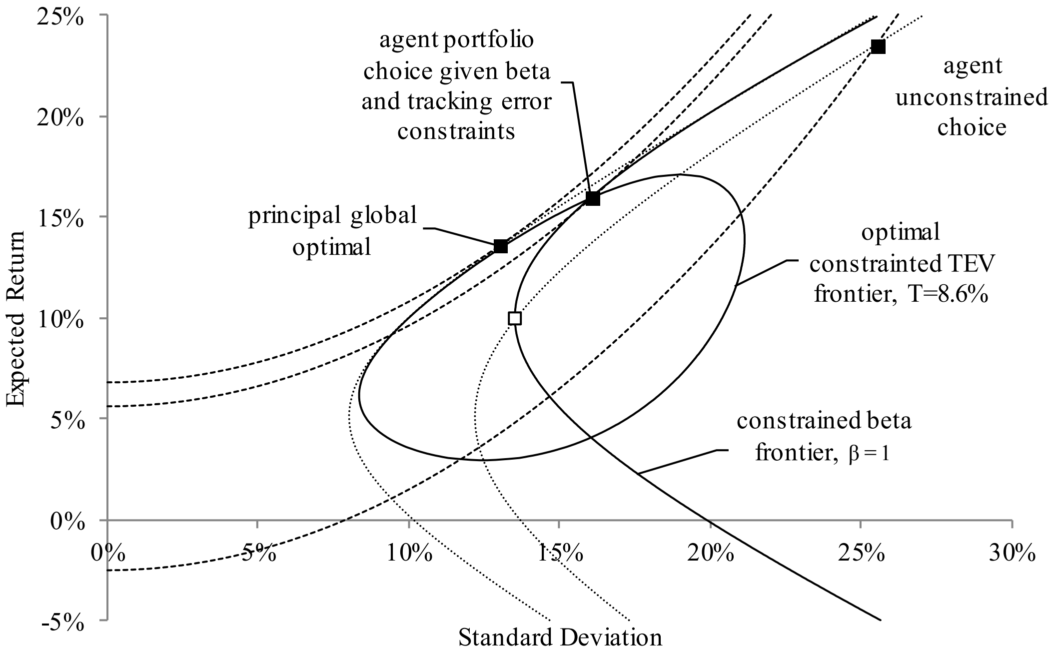


This figure shows various levels of beta and tracking error constrained frontiers along with the principal's optimal constraints. Given the optimal constraints, the agent will choose the principal's global optimal portfolio. It is evident from the figure that the agent's utility is sub-optimal but that the principal's utility is maximized. The optimally constrained beta and tracking error curves and the efficient frontier are all tangent to the principal's utility curve at the agent's choice portfolio.

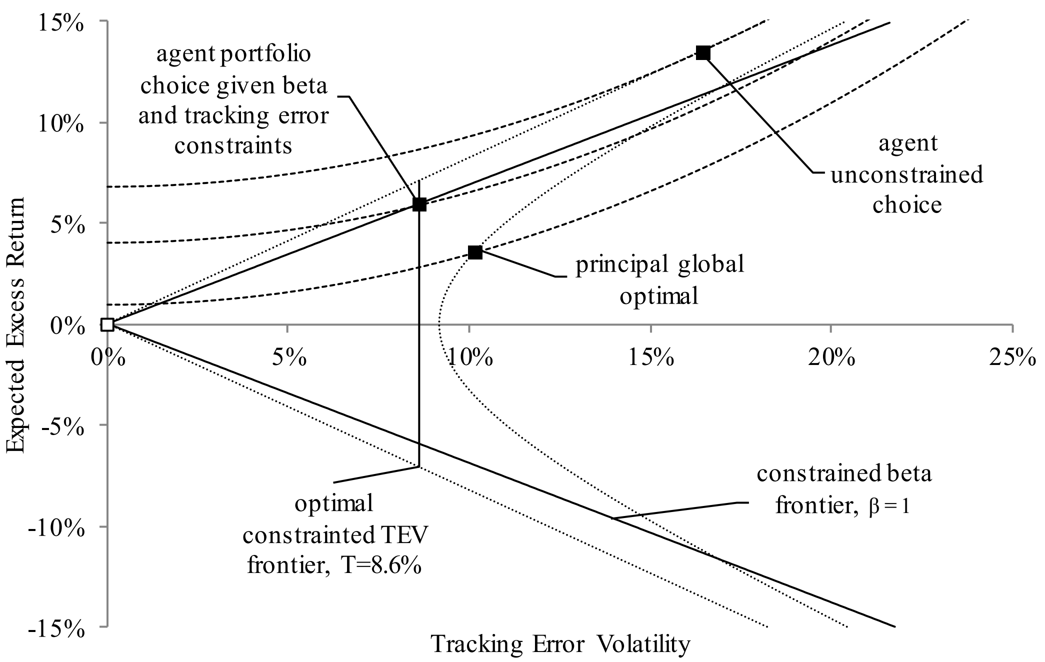
Figure 6

Maximizing Principal Utility by Choosing Tracking Error given a Beta Constraint of 1

Panel A: Mean-Variance Space



Panel B: Mean-TEV Space



In this figure, the principal uses a fixed beta constraint of 1 and chooses a tracking error to maximize his utility given the agent's portfolio choice. The constrained TEV ellipse passes through the point at which the principal's utility is tangent to the constrained beta curve. Also depicted are the agent's unconstrained choice and the principal's global optimal portfolios.

Table 1
Utility Deviations from the Unconstrained Delegated Portfolio Management Performance Incentive

Panel A: Principal's Utility Deviations From Benchmark

Agent Risk Aversion	Principal Risk Aversion Coefficient: 6	5.5	5	4.5	4	3.5	3	2.5	2	1.5
6	-2.09%	-1.45%	-0.80%	-0.16%	**0.48%**	**1.13%**	**1.77%**	**2.42%**	**3.06%**	**3.70%**
5.5	-2.56%	-1.84%	-1.11%	-0.38%	**0.34%**	**1.07%**	**1.79%**	**2.52%**	**3.24%**	**3.97%**
5	-3.19%	-2.36%	-1.53%	-0.70%	**0.13%**	**0.96%**	**1.79%**	**2.62%**	**3.45%**	**4.28%**
4.5	-4.04%	-3.08%	-2.12%	-1.15%	-0.19%	**0.77%**	**1.74%**	**2.70%**	**3.66%**	**4.63%**
4	-5.25%	-4.11%	-2.97%	-1.83%	-0.68%	**0.46%**	**1.60%**	**2.74%**	**3.88%**	**5.03%**
3.5	-7.04%	-5.65%	-4.25%	-2.86%	-1.47%	-0.08%	**1.31%**	**2.70%**	**4.09%**	**5.49%**
3	-9.82%	-8.06%	-6.30%	-4.55%	-2.79%	-1.03%	**0.73%**	**2.48%**	**4.24%**	**6.00%**
2.5	-14.49%	-12.15%	-9.82%	-7.48%	-5.15%	-2.82%	-0.48%	**1.85%**	**4.19%**	**6.52%**
2	-23.18%	-19.84%	-16.50%	-13.16%	-9.82%	-6.48%	-3.14%	**0.20%**	**3.54%**	**6.88%**
1.5	-42.18%	-36.79%	-31.39%	-26.00%	-20.61%	-15.21%	-9.82%	-4.43%	**0.97%**	**6.36%**

Panel B: Principal's Utility Deviations from the Global Optimal Portfolio

Agent Risk Aversion	Principal Risk Aversion Coefficient: 6	5.5	5	4.5	4	3.5	3	2.5	2	1.5
6	-7.18%	-6.19%	-5.26%	-4.39%	-3.62%	-2.98%	-2.54%	-2.43%	-2.88%	-4.45%
5.5	-7.65%	-6.58%	-5.57%	-4.62%	-3.76%	-3.04%	-2.52%	-2.33%	-2.69%	-4.19%
5	-8.28%	-7.10%	-5.98%	-4.93%	-3.97%	-3.15%	-2.53%	-2.23%	-2.49%	-3.88%
4.5	-9.13%	-7.83%	-6.57%	-5.39%	-4.29%	-3.34%	-2.58%	-2.15%	-2.27%	-3.53%
4	-10.34%	-8.86%	-7.42%	-6.06%	-4.79%	-3.65%	-2.72%	-2.10%	-2.05%	-3.13%
3.5	-12.13%	-10.39%	-8.71%	-7.10%	-5.57%	-4.19%	-3.00%	-2.14%	-1.84%	-2.67%
3	-14.91%	-12.81%	-10.76%	-8.78%	-6.89%	-5.14%	-3.59%	-2.36%	-1.70%	-2.16%
2.5	-19.58%	-16.90%	-14.27%	-11.72%	-9.25%	-6.93%	-4.80%	-2.99%	-1.75%	-1.63%
2	-28.27%	-24.59%	-20.96%	-17.39%	-13.92%	-10.59%	-7.45%	-4.64%	-2.39%	-1.27%
1.5	-47.27%	-41.53%	-35.85%	-30.23%	-24.71%	-19.32%	-14.14%	-9.27%	-4.97%	-1.80%

This table calculates the quatradic utility deviations for the principal under differing scenarios. Panel A depicts the utility increase or decrease for the principal from unconstrained delegated contracting. The delegated performance incentive has the potential to increase utility for the principal but this is far from certain. Given the levels of utility I use in the figures for this paper, the utility decreases by 5.15% through delegation. Panel B shows the utility depreciation from the principal's global optimal portfolio. This is necessarily non-positive. In this paper, the agent chooses a portfolio that 9.25% less optimal in utility than the principal's global optimal.

Table 2
Utility Deviations from TEV Constrained Delegated Portfolio Management

Panel A: Principal's Utility Deviations from the Benchmark given a Fixed Constraint of 8%

		Principal Risk Aversion Coefficient									
		6	5.5	5	4.5	4	3.5	3	2.5	2	1.5
Agent Risk Aversion	6	-2.99%	-2.20%	-1.40%	-0.60%	0.20%	1.00%	1.79%	2.59%	3.39%	4.19%
	5.5	-2.99%	-2.20%	-1.40%	-0.60%	0.20%	1.00%	1.79%	2.59%	3.39%	4.19%
	5	-2.99%	-2.20%	-1.40%	-0.60%	0.20%	1.00%	1.79%	2.59%	3.39%	4.19%
	4.5	-2.99%	-2.20%	-1.40%	-0.60%	0.20%	1.00%	1.79%	2.59%	3.39%	4.19%
	4	-2.99%	-2.20%	-1.40%	-0.60%	0.20%	1.00%	1.79%	2.59%	3.39%	4.19%
	3.5	-2.99%	-2.20%	-1.40%	-0.60%	0.20%	1.00%	1.79%	2.59%	3.39%	4.19%
	3	-2.99%	-2.20%	-1.40%	-0.60%	0.20%	1.00%	1.79%	2.59%	3.39%	4.19%
	2.5	-2.99%	-2.20%	-1.40%	-0.60%	0.20%	1.00%	1.79%	2.59%	3.39%	4.19%
	2	-2.99%	-2.20%	-1.40%	-0.60%	0.20%	1.00%	1.79%	2.59%	3.39%	4.19%
	1.5	-2.99%	-2.20%	-1.40%	-0.60%	0.20%	1.00%	1.79%	2.59%	3.39%	4.19%

Panel B: Principal's Utility Deviations from the Benchmark given an Optimal Tracking Error Constraint

		Principal Risk Aversion Coefficient									
		6	5.5	5	4.5	4	3.5	3	2.5	2	1.5
Agent Risk Aversion	6	0.05%	0.12%	0.25%	0.45%	0.74%	1.17%	1.80%	2.74%	4.26%	6.90%
	5.5	0.05%	0.12%	0.25%	0.45%	0.74%	1.17%	1.80%	2.74%	4.26%	6.90%
	5	0.05%	0.12%	0.25%	0.45%	0.74%	1.17%	1.80%	2.74%	4.26%	6.90%
	4.5	0.05%	0.12%	0.25%	0.45%	0.74%	1.17%	1.80%	2.74%	4.26%	6.90%
	4	0.05%	0.12%	0.25%	0.45%	0.74%	1.17%	1.80%	2.74%	4.26%	6.90%
	3.5	0.05%	0.12%	0.25%	0.45%	0.74%	1.17%	1.80%	2.74%	4.26%	6.90%
	3	0.05%	0.12%	0.25%	0.45%	0.74%	1.17%	1.80%	2.74%	4.26%	6.90%
	2.5	0.05%	0.12%	0.25%	0.45%	0.74%	1.17%	1.80%	2.74%	4.26%	6.90%
	2	0.05%	0.12%	0.25%	0.45%	0.74%	1.17%	1.80%	2.74%	4.26%	6.90%
	1.5	0.05%	0.12%	0.25%	0.45%	0.74%	1.17%	1.80%	2.74%	4.26%	6.90%

This table calculates the quadratic utility deviations for the principal under differing scenarios. Panel A depicts the utility increase or decrease for the principal from a fixed tracking error constraint of 8%. The delegated performance incentive with a tracking error constraint has the potential to increase utility for the principal if the constraint is chosen appropriately. Given the levels of utility I use in the figures for this paper, the utility increases by 0.20% through delegation and an 8% constraint. Panel B shows the utility increase from the benchmark given an optimal tracking error constraint; this is necessarily non-negative. Also depicted is that the level of utility increase is independent of agent risk aversion.

Table 3
Utility Deviations from Beta Constrained Delegated Portfolio Management

Panel A: Principal's Utility Deviations from the Benchmark given a Fixed Beta Constraint of 1

		Principal Risk Aversion Coefficient									
		6	5.5	5	4.5	4	3.5	3	2.5	2	1.5
Agent Risk Aversion	6	1.98%	2.14%	2.31%	2.47%	2.64%	2.80%	2.97%	3.13%	3.30%	3.46%
	5.5	1.96%	2.16%	2.35%	2.55%	2.75%	2.94%	3.14%	3.34%	3.53%	3.73%
	5	1.90%	2.14%	2.37%	2.61%	2.85%	3.09%	3.32%	3.56%	3.80%	4.04%
	4.5	1.76%	2.05%	2.34%	2.64%	2.93%	3.22%	3.52%	3.81%	4.10%	4.40%
	4	1.48%	1.85%	2.23%	2.60%	2.97%	3.34%	3.71%	4.08%	4.45%	4.82%
	3.5	0.97%	1.45%	1.94%	2.42%	2.91%	3.39%	3.88%	4.36%	4.85%	5.33%
	3	0.00%	0.66%	1.32%	1.98%	2.64%	3.30%	3.96%	4.62%	5.28%	5.94%
	2.5	-1.90%	-0.95%	0.00%	0.95%	1.90%	2.85%	3.80%	4.75%	5.70%	6.65%
	2	-5.94%	-4.45%	-2.97%	-1.48%	0.00%	1.48%	2.97%	4.45%	5.94%	7.42%
	1.5	-15.83%	-13.19%	-10.55%	-7.91%	-5.28%	-2.64%	0.00%	2.64%	5.28%	7.91%

Panel B: Principal's Utility Deviations from the Benchmark given an Optimal Beta Constraint

		Principal Risk Aversion Coefficient									
		6	5.5	5	4.5	4	3.5	3	2.5	2	1.5
Agent Risk Aversion	6	5.09%	4.73%	4.39%	4.07%	3.77%	3.52%	3.33%	3.23%	3.30%	3.70%
	5.5	5.07%	4.75%	4.44%	4.15%	3.88%	3.66%	3.50%	3.43%	3.53%	3.97%
	5	5.01%	4.72%	4.46%	4.21%	3.99%	3.80%	3.68%	3.66%	3.80%	4.28%
	4.5	4.87%	4.64%	4.43%	4.23%	4.07%	3.94%	3.88%	3.91%	4.11%	4.64%
	4	4.59%	4.44%	4.31%	4.19%	4.10%	4.06%	4.07%	4.18%	4.45%	5.06%
	3.5	4.08%	4.04%	4.02%	4.02%	4.04%	4.11%	4.24%	4.46%	4.85%	5.57%
	3	3.11%	3.25%	3.40%	3.57%	3.77%	4.02%	4.32%	4.71%	5.28%	6.18%
	2.5	1.21%	1.64%	2.08%	2.54%	3.04%	3.57%	4.16%	4.84%	5.70%	6.89%
	2	-2.83%	-1.86%	-0.89%	0.11%	1.14%	2.20%	3.33%	4.55%	5.94%	7.66%
	1.5	-12.72%	-10.60%	-8.47%	-6.32%	-4.14%	-1.92%	0.36%	2.73%	5.28%	8.16%

This table calculates the quadratic utility deviations for the principal under differing scenarios. Panel A depicts the utility increase or decrease for the principal from a fixed Beta constraint of 1. The delegated performance incentive with a beta constraint has the potential to increase utility for the principal if the constraint is chosen appropriately. Given the levels of utility I use in the figures for this paper, the utility increases by 1.90% through delegation and a beta of 1. Panel B shows the utility increase from the benchmark given an optimal beta constraint; this is necessarily better than Panel A. Note that the level of utility increase is substantial and relatively certain under reasonable levels of risk aversion.

Table A1
Numerical Values Used to Build the Figures and Tables

r
0.06
0.10
0.13
0.04

b
0
1
0
0

Ω			
0.01114	0.00639	0.00522	0.00127
0.00639	0.01836	0.00948	0.00843
0.00522	0.00948	0.01779	0.00676
0.00127	0.00843	0.00676	0.01245

Every figure and calculation done in this paper were built with these three matrices. I tried to be a pure as I could in the expressions without redefining variables, as much as possible. These values were chosen somewhat arbitrarily for ease of depiction and exposition. The final examples are mildly representative of a realistic scenario but not actually done with real calculations.